Arithmetic and Algebra as an Independent Development

OLLI Summer 2014

Fractions

- Babylonian and Egyptians used fractions
- Classical Greek Mathematicians used ratios
  - Numbers were lengths
- Fractions used in Greek commerce
- Alexandrian Mathematicians used fractions in there own right
  - Some used the Egyptian system
  - Some used the accent system
    - 13/29 ιγ′ κθ″
  - Some used the Babylonian base 60 system
    - The nonpositional alphabetic numerals for the whole numbers and the base 60 system of place value for the fractions
    - Ptolemy's Almagest (c 150 CE)

Square Root

- Babylonians – yes $\sqrt{2}$
- Pretty much ignored by the Classical Greeks because of the difficulty with irrational numbers
- Archimedes has $1351/780 > \sqrt{3} > 265/153$ which is pretty accurate, but he does not tell how he got it.
- Heron aka Hero of Alexandria (10 – 75 CE) uses

  $\sqrt{A} = \sqrt{a^2 \pm b} \sim a \pm b/(2a)$ where a^2 is the rational square nearest A and b is the remainder. This expression was also used by the Babylonians. It is also the first two terms of Maclaurin series. Heron also has a rule for calculating the cube root.

Heron (10 – 75 CE)

- Heron (10 – 75 CE), Nicomachus (60 – 120 CE), and Diophantus (200 – 284 CE) treated arithmetical and algebraic problems in and for themselves and did not depend upon geometry for motivation or to bolster the logic (M. Kline).
- Heron solves quadric equations the way the Babylonians did. From *Geometrica:* Given a square such that the sum of the area and perimeter is 896. $x^2 + 4x = 896$. Heron takes half of 4 and adds its square, completing the square on the left side.
- Indeterminate problems also are in the *Geometrica.*
- Heron was an engineer who worked in many field and a large number of his writings have survived. He is credited with inventing the first vending machine.


directional nozzle
steam exhaust
causes sphere to spin
pivot
steam rises
through tubes
water vaporized
in heated kettle

Nicomachus (60 – 120 CE)

- A Pythagorean
- Wrote an Introduction to Arithmetic which was the first work to treat arithmetic as a separate topic from geometry.
- Most of the work was not original. Some theorems proved in the Elements.
- No proof of theorems, merely states and illustrates with examples.
- Some of the theorems were wrong – the results just happened to be correct for the cases he chose.
  - Nicomachus states that the *n*th perfect number has n digits, and that all perfect numbers end in 6 or 8
  - 6, 28, 496, 8128 known to Greeks
  - 33550336 (discovered in 1456 CE)

Nicomachus (60 – 120 CE)

- One interesting contribution was the observation that if one writes down the odd numbers

  1, 3, 5, 7, 9 ,11, 13, 15, 17, …

  The first is the cube of 1, the sum of the next 2 (3 + 5) is the cube of 2, the sum of the next 3 (7 + 9 + 11) is the cube of 3, and so on.
- The *Introduction to Arithmetic* includes the multiplication table for the numbers 1 to 9 precisely as we learn it.
- It had value because it was a systematic, orderly, clear, and comprehensive presentation of the arithmetic of integers and ratios of integers freed of geometry.
- It became the standard test in arithmetic for the next 1000 years. After Nicomachus arithmetic became the ***in*** subject at Alexandria.

Diophantus of Alexandria

Diophantus of Alexandria

- Author of the **Arithmetica** one of the greatest mathematical treatises of ancient times
- Generally believed he lived in the 3rd century CE.
  - Not mentioned by Nicomachus (c. 100) nor Theon of Smyrna (c. 130)
  - Quoted by Theon of Alexandria (c. 365)
  - The commentary on his work by Hypatia (c. 415) is the ultimate source of all existing manuscripts and translations of the **Arithmetica**
- The Arithmetica was so thorough and complete a treatment of algebraic analysis in its time, that all previous works in its field ceased to be of interest and disappeared.

Hypatia (c. 370 – 415 CE)

- **Hypatia of Alexandria** was the first woman recognized as a mathematician since the recording of history. Her interests included astronomy, philosophy, and inventions. In 415 Hypatia was tortured to death by religious zealots following the new Christian patriarch Cyril of Alexander.
- *"Fable should be taught as fable, myth as myth, and miracles as poetic fancies. To teach superstitions as truth is horrifying. The mind of a child accepts them and only through great pain, perhaps tragedy, can the child be relieved of them. Men will fight for superstition as quickly as for the living truth -- even more so, since a superstition is intangible, you can't get at it to refute it, but truth is a point of view, as so is changeable."*

Epitaph

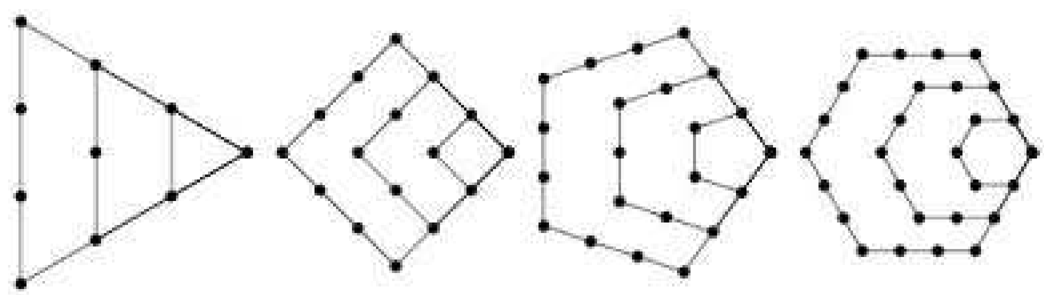
- This tomb holds Diophantus. Ah, what a marvel!
  And the tomb tells scientifically the measure of his life.
  God vouchsafed that he should be a boy for the sixth part of his life;
  when a twelfth was added, his cheeks acquired a beard;
  He kindled for him the light of marriage after a seventh,
  and in the fifth year after his marriage He granted him a son.

  Alas! late-begotten and miserable child,
  when he had reached the measure of half his father's life,
  the chill grave took him.
  After consoling his grief by this science of numbers for four years,
  he reached the end of his life.

  Metrodorus – Greek Anthology (c. 500 CE)

Works of Diophantus

- The Arithmetica (originally in 13 Books – only 6 survive)
  - The missing books were probably lost at a very early date because the are not included in Hypatia's commentary.
- A tract on Polygonal Numbers (only a fragment survives)



- A collection of propositions under the title of Porisms
  - Contains some propositions on the theory of numbers

Algebra - the Three Stages

- Stage 1
  - Prior to Diophantus' time, the steps in solving a problem were written in words and complete sentences, like a piece of prose, or a philosophical argument.
- Stage 2
  - Similar to the first stage, but marked by the use of abbreviational symbols for constantly recurring quantities and operations.
  - Includes Diophantus and all the later Europeans until the middle of the 17th century except for Francois Viete (1540 – 1603), who used letters to represent both constants and variables.
- Stage 3
  - Uses a complete system of notation by symbols having no visible connection with the words or things they represent.

Greek Numbers

1 = α	10 = ι	100 = ρ
2 = β	20 = κ	200 = σ
3 = γ	30 = λ	300 = τ
4 = δ	40 = μ	400 = υ
5 = ε	50 = ν	500 = φ
6 = ς (ϝ)	60 = ξ	600 = χ
7 = ζ	70 = ο	700 = ψ
8 = η	80 = π	800 = ω
9 = θ	90 = ϙ	900 = ϡ

1,000	͵α		χίλιοι, -αι, -α
2,000	͵β		δισχίλιοι
3,000	͵γ		τρισχίλιοι
10,000	͵ι	$\overset{\text{Α}}{\text{M}}$	μύριοι, -αι, -α
20,000	͵κ	$\overset{\text{Β}}{\text{M}}$	δισμύριοι
100,000	͵ρ		δεκακισμύριοι

Diophantus's Notation

- ἀριθμός (number) denoted by ς (x) used for the unknown
- δύναμις (power) denoted by Δ^{Y} (x^2) used for the square of the unknown
- κύβος denoted by K^{Y} (x^3) used for the cube of the unknown and numbers
- δυναμοδύναμις by $\Delta^{Y}\Delta$ (x^4)
- δυναμόκυβος by ΔK^{Y} (x^5)
- κυβόσκυβος by $K^{Y}K$ (x^6)
- ἀριθμοστόν by **ς**$^{\chi}$ ($1/x$), $\Delta^{Y\chi}$ ($1/x^2$)
- ΔK^{Y} κ ζ = $27x^5$, $\Delta^{Y\chi}$ σν = $250/x^2$

Diophantus's Notation (continued)

- K^Y α Δ^Y ι γ**ς**ε = $x^3 + 13x^2 + 5x$
  - No signs are needed for multiplication and division
  - Addition is indicated by juxtaposition
  - Subtraction is indicated by ⩚
  - When there are units in addition, the units are indicated by $\overset{\circ}{M}$
  - $5x + 2$ = **ς**ε $\overset{\circ}{M}$ β

The Arithmetica (189 problems)

Deals with problems solved by determinate equations (DE) and indeterminate equations (IE)

Book 1 consists mainly of problems that lead to DE of the first degree. The remaining 5 books treat mainly IE of the second degree.

DE are equations where there the solutions are uniquely determined. Example: $x^2 = 4$ yields $x = 2$ (note no -2)

IE usually have fewer condition than unknowns and there are an infinite number of solutions. Example: Divide a given square number into two squares. If the given square number is 4^2 some solutions are $(16/5)^2 + (12/5)^2$ & $(20/13)^2 + (48/13)^2$

Book 1 – Pure Determinate Equations

- Diophantus gives a general rule for this case without regard to degree.

  We have to take like from like on both sides of an equation and neutralize negative terms by adding to both sides, then take like from like again, until we have one term left equal to one term.

- After these operations have been performed, the equation (after dividing out, if both sides contain a power of x, by the lesser power) reduces to $Ax^m = B$, and is considered solved.
  - Only one root
  - No negative roots or zero
  - Only rational solutions
  - If there are two positive roots – the solution is the largest root

Book 1 Problem 8

To two given numbers to add the same number so as to make the resulting numbers have to one another a given ratio.

Solution: Given numbers 100, 20, given ratio 3:1

Required number x. $(100 + x):(20 + x) = 3:1$

Therefore $3x + 60 = x + 100$

$$3x - x + 60 = x - x + 100$$

$$2x + 60 = 100$$

$$2x + 60 - 60 = 100 - 60$$

$$2x = 40$$

$$x = 20$$

The balance scale approach.

Book 1 Problem 18

To find three numbers such that the sum of any pair exceeds the third by a given number.

Solution: Given excesses 20, 30, 40. Let 2x = sum of all three numbers

(1) + (2) = (3) + 20, add (3) to each side (1) + (2) + (3) = 2(3) + 20,

2x = 2(3) + 20 or (3) = x - 10

(2) + (3) = (1) + 30, add (1) to each side x = (1) + 15, (1) = x - 15

(1) + (3) = (2) + 40, (2) = x – 20

2x = 3x-45

x = 45

(1) = 30, (2) = 25, (3) = 35

Diophantus' Method

- Quadratic Equations
  - For $ax^2 + bx + c = 0$ we know that $x = (- b \pm \sqrt{(b^2 – 4ac)})/(2a)$
  - Diophantus thought in terms of three equations
  - (a) $mx^2 + px = q$, (b) $mx^2 = px + q$, (c) $mx^2 + q = px$

  Solutions are:

  (a) $(-\frac{1}{2} p + \sqrt{(\frac{1}{4}p^2 + mq)})/m$

  (b) $(\frac{1}{2} p + \sqrt{(\frac{1}{4}p^2 + mq)})/m$

  (c) $(\frac{1}{2} p + \sqrt{(\frac{1}{4}p^2 - mq)})/m$
- In those cases where the √ was irrational, "D" new how to approximate it by a rational number

Simultaneous equations involving quadratics

- Book 1 problem 30
- To find two numbers such that their difference and product are given numbers.
- Difference = 4 and product 96.
- This is a standard format problem and it would immediately be said that the numbers are $x + 2$ and $x – 2$.
- The equation to meet the product requirement is $(x + 2)(x – 2) = 96$

$$x^2 - 4 = 96$$
$$x^2 = 96 + 4$$
$$x^2 = 100$$
$$x = 10$$ and the two numbers are 12 and 8.

Book II, Problem 8

- To divide a given square number into two squares

It is to this proposition that Fermat appended his famous note in which he enunciates what is known as the "great theorem" of Fermat. The text of the note is as follows :

"On the other hand it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or generally any power except a square into two powers with the same exponent. I have discovered a truly marvelous proof of this, which however the margin is not large enough to contain."

Did Fermat really possess a proof of the general proposition that $x^m + y^m = z^m$ cannot be solved in rational numbers where m is any number >2?

Fermat’s Last theorem

- In 1908, the German industrialist and amateur mathematician Paul Wolfskehl bequeathed 100,000 marks to the Göttingen Academy of Sciences to be offered as a prize for a complete proof of Fermat's Last Theorem. On 27 June 1908, the Academy published nine rules for awarding the prize. Among other things, these rules required that the proof be published in a peer-reviewed journal; the prize would not be awarded until two years after the publication; and that no prize would be given after 13 September 2007, roughly a century after the competition was begun. British mathematician Andrew Wiles collected the Wolfskehl prize money, then worth $50,000, on 27 June 1997.

Indeterminate Equations

- Since Diophantus allowed fractional as well as integer solutions to IEs, it was not necessary to treat equations of the first degree
- Equations of the second degree
  - $Ax^2 + Bx + C = y^2$
- Double equations
  - Two different functions of the unknown have to be made simultaneously squares.
  - $mx^2 + \alpha x + a = u^2$ and $nx^2 + \beta x + b = w^2$
- Simple example - $m = n = 0$, find x so u and w are rational.

  Eq (1) $65 - 6x = u^2$ and Eq (2) $65 - 24x = w^2$

  4 times Eq(1) – Eq(2) yields $195 = 4u^2 - w^2$, $15 \cdot 13 = (2u + w)(2u - w)$

  Let $2u + w = 15$ and $2u - w = 13$, then $u = 7$, $w = 1$, and $x = 8/3$.

Book IV Problem 22 p1

To find three numbers (a, b, and c) such that their solid content (sc) added to any one of them gives a square.

- The solid content is the product of the three numbers.
- By a square is meant $(p/q)^2$ where p and q are integers (p/q is rational)
- Remember only one variable can be used as an unknown in an equation.
- The problem has multiple answers but "D" is only trying to find one of them.

Book IV Problem 22 p2

1. Assume the sc is of the form $x^2 + 2x$
2. Assume the first number (a) is 1 then sc + a = $(x+1)^2$ and as long as x is rational, sc + a is a square.
3. Assume the second number (b) is $4x + 9$. If this works, then sc + b = $x^2 + 6x + 9$ or $(x + 3)^2$
4. The third number (c) is sc/(ab) = $(x^2 + 2x)/(4x + 9)$
5. It is not easy to see how to pick a value of x so that sc + c is a square.
6. Go back to step 3 and generalize the expression for b so that sc + b = $(x + m)^2$, then b = $2mx – 2x +m^2$

Book IV Problem 22 p3

7. $c = (x^2 + 2x)/(2mx - 2x + m^2)$
8. "D" now observes if $m = 2$, then $c = x/2$ and $b = 2x + 4$
9. $sc + c = x^2 + 2x + x/2 = x^2 + 5x/2$
10. To ensure $sc + c$ is a square, "D" sets $x^2 + 5x/2 = 4x^2$ the solution of which is $x = 5/6$.
11. The three numbers are 1, 17/3, and 5/12

If I had chosen $x^2 + 5x/2 = (p/q)^2x^2$ instead of $4x^2$, then $x = 5q^2/\{2(p^2 - q^2)\}$ and there are solutions for all integers p, q with $p > q$.

Fermat's Solution

- Fermat (1601 – 1665)
- Let the sc = $x^2 - 2x$ and the first number equal 1
- Let the second number equal 2x
- The third number = sc/(ab) = $(x^2 - 2x)/(2x) = (x - 2)/2$ and sc + c = $x^2 - 2x + x/2 - 1 = x^2 - (3/2)x - 1$ must be a square.
- x must be > 2
- $x^2 - (3/2)x - 1 = (x - m)^2$, $x = (m^2 + 1)/(2m - 1.5)$
- If m chosen such that x > 2, a solution is obtained.

Porisms

– Porism # 3 The difference of any two cubes is also the sum of two cubes. Given a and b, $a > b$, then there exists an x and y such that

$$a^3 – b^3 = x^3 + y^3$$

Diophantus does not give the solution, but one is provided by 16th century French mathematician François Viète.

$$x = a(a^3 – 2b^3)/(a^3 + b^3), \quad y = b(2a^3 – b^3)/(a^3 + b^3)$$

Example: $a =4$, $b =3$, $x = 40/91$, $y = 303/91$

Book VI, Problem 17

- To find a right-angled triangle such that the area added to the hypotenuse gives a square, while the perimeter is a cube.
- It is shown that a square must be found that, when 2 is added to it, becomes a cube. The solution for the sides of the triangle is 2 621/50 629/50, the square 25, and cube 27
- Fermat (1601 – 1665) comments that the equations $u^2 + 2 = v^3$ has only one solution u = 5 and v = 3.
- A proof is provided by Euler (1707- 1783)

Diophantus of Alexandria : a Text and its History
Norbert Schappacher April 2005

- $y^2 = x^6 + x^2 + 1$
- Blowing up at infinity resolves this into a smooth projective curve of genus 2. According to Gerd Faltings's theorem from 1983 (the former Mordell Conjecture), such a curve can only have finitely many rational points.
- Thus Wetherell writes in his introduction:
- This work was motivated by a problem from the Arithmetica of Diophantus. In problem 17 of book 6 of the Arabic manuscript, Diophantus poses a problem which comes down to finding positive rational solutions to $y^2 = x^6 + x^2 + 1$. This equation describes a genus 2 curve which we will call C. Diophantus provides the solution (1/2, 9/8) and a natural question is whether there are any other positive rational solutions. It clearly will suffice to find all rational points on C. In addition to the solution given by Diophantus … .

Summary

- Introduce symbolic notation
- Great skill in reducing equations to forms he can handle
- No general methods, no deductive logic as in Greek geometry
- When solving quadratic equations, he accepts only positive rational roots and when there are two positive roots he accepts only the larger.
- Uses no geometry
- Determinate equations went no further than the Babylonians
- Diophantus’ work on indeterminate equations appears to be new and work in that field today bears his name

Epitaph

- “D” lived to x years
- Boyhood lasted 1/6 of his life $x/6$
- Grew a beard after another 1/12 $x/12$ $x/6 + x/12 = x/4$
- After 1/7 more he married $x/7$ $x/4 + x/7 = 11x/28$
- Had a son 5 years later $11x/28 + 5$
- Son lived to half fathers age $11x/28 + 5 + x/2 = 25x/28 + 5$
- “D” died 4 years later

$25x/28 + 5 + 4 = x$, $3x/28 = 9$, $x = 84$

References

- The Treasury of Mathematics by Henrietta O. Midonick
- Mathematical Thought from Ancient to Modern Times by Morris Kline
- A History of Mathematics by Carl Boyer
- A History of Greek Mathematics by Sir Thomas Heath
- Diophantus of Alexandria; A study in the History of Greek Algebra by Sir Thomas Heath

History of Mathematics Time-Line
William H. Richardson, Wichita State U

- 60 Geminus on the parallel postulate
- 75 Works of Heron of Alexandria
- 100 Menelaus' *Spherica*
- 150 Ptolemy's *Almagest*
- 250 Diophantus' *Arithmetica*
- 320 Pappus' *Mathematical Collections*
- 415 Death of Hypatia
- 529 Closing of the schools at Athens
- 641 Library at Alexandria burned
- 775 Hindu works translated into Arabic
- 830 Al-Khowarizmi's *Algebra* (Arabic)
- 1114 Birth of Bhaskara (Indian)
- 1202 C.E. Fibonacci's *Liber abaci*
- 1303 Chu Shih-Chieh and the Pascal triangle (long before Pascal)
- 1464 Death of Nicolas of Cusa
- 1489 Use of + and – by Widmann
- 1492 Use of decimal point by Pellos
- 1527 Apian publishes the Pascal triangle
- 1544 Stifel publishes *Arithmetica integra*
- 1545 Cardan publishes *Ars magna*
- 1564 Birth of Galileo
- 1572 Bombelli's *Algebra*
- 1579 Viète publishes *Canon mathematicus*
- 1595 Pitiscus publishes *Trigonometria*
- 1609 Kepler's *Astronomia nova* : Galileo's telescope
- 1614 Napier's logarithms
- 1620 Bürgi's logarithms
- 1629 Fermat's method of maxima and minima
- 1637 Descartes' *Discours de la méthode*