Conics

OLLI Summer 2014

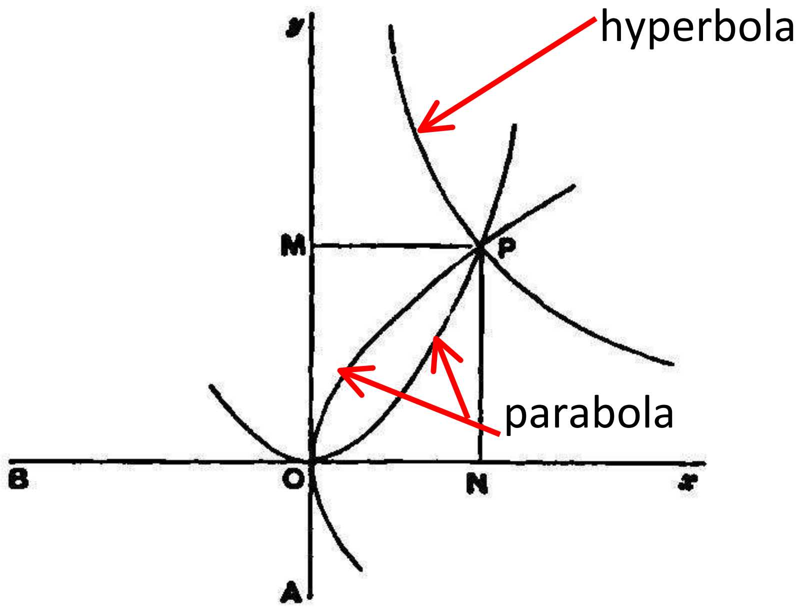
Menaechmus

- Born: about 380 BC in Alopeconnesus, Asia Minor (now Turkey), Died: about 320 BC
- Pappus (290 –350 CE) and Proclus (412–485 CE)
- A student of Eudoxus and a contemporary of Plato
- Discover of the conic sections
- Solved the problem of the two mean proportionals (Hippocrates - duplication of the cube) by means of conic sections.
- Is credited with the discovery of the parabola, rectangular hyperbola, and ellipse

Two Mean Proportionals

- Assume that a, b are two given unequal straight lines and x, y the two required mean proportionals, the discovery of Hippocrates amounted to the fact that from the relation
- $(a/x) = (x/y) = (y/b)$
- If follows that $x^2 = ay$, $y^2 = bx$, and $xy = ab$
- And $(a/x)^3 = (a/b)$
- If $b = 2a$, $2a^3 = x^3$
- The solutions of Menaechmus amount to a solution of any two of the equations.

One Solution



It is believed that the discovery by “M” was not the result of a systematic study of the sections of a cone. More likely he looked for ways to generate these figures and found he could get them from a right cone.

Apollonius gave the conic sections the names we know them by.
Menaechmus called a parabola a section of a right-angled cone , an hyperbola a section of an obtuse-angle cone, and an ellipse a section of an acute angled cone.
Only cones with vertex above the center of the circular base.

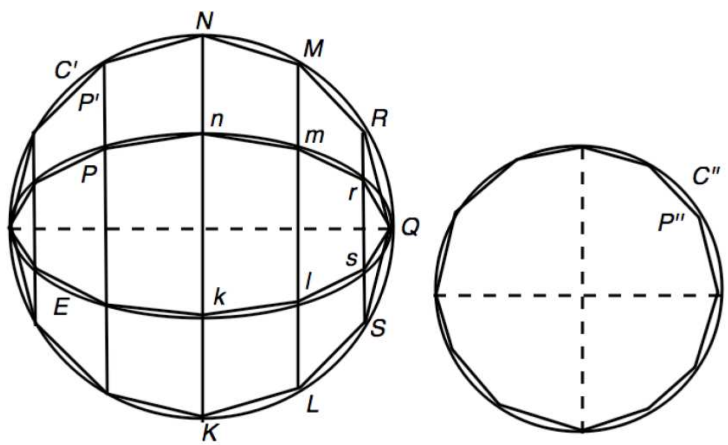
Conics After Menaechmus

- Treatises by Aristaeus the elder (active 370 BCE-300 BCE)
- Euclid – Four books on conics which were lost. Probably a compilation of earlier works. Believe they are the first four books of Apollonius’s eight books on conics.
- Archimedes
  - Heracleides, the biographer of Archimedes, is quoted as saying that Archimedes was the first to invent certain theorems in conics, and that Apollonius, having found that they had not been published by Archimedes, took credit for them.
  - Eutocius responded that the allegation is in his opinion not true, “for on one hand Archimedes appears in many passages to have referred to the elements of conics as an older treatise, and on the other hand Apollonius does not profess to be giving his own discovers.”

Archimedes and Conics

- Basic results from Euclid and others used without proof.
  - The straight line drawn from the center of an ellipse, or point of intersection of the asymptotes of a hyperbola, through the point of contact of any tangent, bisects all chords parallel to the tangent.
  - In the ellipse the tangents at the extremities of either of the two conjugate diameters are both parallel to the other diameter
  - If a line between the asymptotes meets a hyperbola and is bisected at the point of concourse, it will touch the hyperbola.
  - In a hyperbola, if P be any point on the curve and PK, PL are straight lines drawn parallel to the asymptotes and meeting the other, the rectangle $PK \cdot PL$ is a constant.

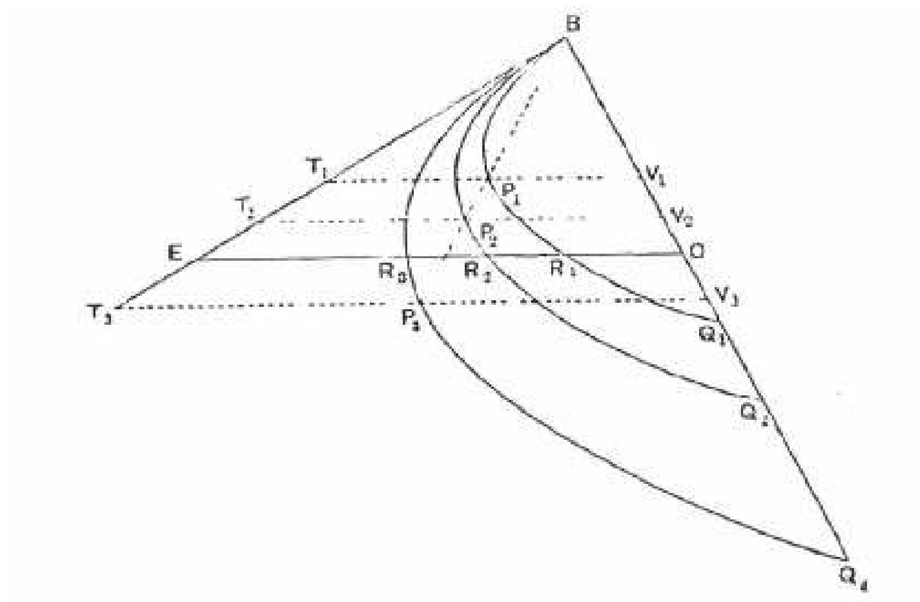
Area of an Ellipse



E is an ellipse with major axis “a” and minor axis “b”.
C’ is a circle with radius “a” circumscribed about E.
C” is a circle with radius $\sqrt{}$ab.
Archimedes proved that the area of E equals the area of C” using the method of exhaustion.
The area of C” = πab.
Note there is no simple rule for the length of the circumference of an ellipse.

This is proposition 4 in On Conoids and Spheroids

Archimedes Example



If any three similarly situated parabolic segments have one extremity (B) of their bases common and their bases BQ_1, BQ_2, BQ_3 lying along the same straight line, and if EO be drawn parallel to the axis of any of the segments meeting the tangent at B to one of them in E, the common base in O, and each of the three segments in R_1, R_2, R_3 then

$$\frac{R_3R_2}{R_2R_1} = \frac{Q_2Q_3}{BQ_3} \bullet \frac{BQ_1}{Q_1Q_2}$$

Bio Apollonous
Devoted to Apollo

- Born about 262 BCE in Perga, Pamphylia, Greek Ionia
  - Now Murtina, Antalya, Turkey
- As a young man he went to Alexandria where he studied under the followers of Euclid and he later taught there.
- In the preface to one of his books he notes that he has a son also named Apollonous.
- His famous work is his 8 books on Conics. Only the first 4 survive in Greek. Books 5 - 7 survive in Arabic.
- In 1710 Halley provided a Latin translation of books 1 – 7.

Bio continued

- Most of the results in Books 1 -4 were known to Euclid and others
- Books 5 – 7 were highly original
- Remnants of Book 8 were used to partially reconstruct it
- Pappus (290 – 350 CE) identifies six other works
  - Cutting-off of a ratio, Cutting an area, On determinate section, Tangencies, Plane loci, and On verging constructions
  - Cutting of a ratio survives in Arabic
- We know from other commentaries of other lost works
  - Regular solids, Theory of irrationals, on a number system, on properties of burning mirrors
  - In his work on burning mirrors Apollonius showed that light from the sun is not focused by a spherical mirror but a parabolic mirror
- Apollonius died at the age of ~ 72 in Alexandria (190 BCE)

Some of the Lost Works

- Tangencies
  - Contains the famous Apollonian problems: Given any three points, lines, or circles, or any combination of three of these, to construct a circle passing through the points and tangent to the given lines or circles.
  - Early mathematicians including Newton were under the impression that Apollonius had not solved the 3 circle case and took it as a challenge.
- Simple cases of Apollonian problems
  - 3 points and 3 lines
  - Slightly more complicated 2 lines and 1 point & 2 points and a line

Apollonius' Conics

- The contents of the eight books of the *Conics* are described in the prefaces to books and are contained in the letter that Apollonius sent with each book.
- Book 1: Apollonius to Eudemus, greetings. If you are in good health and things are in other respects as you wish, it is well; with me too things are moderately well. During the time I spent with you at Pergamum I observed your eagerness to become acquainted with my work on conics; I am therefore sending you the first book, which I have corrected, and will forward the remaining books when I have finished them to my satisfaction. …

Apollonius' Conics – Book 1 Preface Cont.

Now of the eight books the first four form an elementary introduction. The first contains the modes of producing the three sections and the opposite branches (of the hyperbola) and the fundamental properties subsisting in them, worked out more fully and generally than in the writings of others.

The second book contains the properties of the diameters and axes of the sections as well as the asymptotes ….

The third book contains many remarkable theorems useful for the syntheses of solid loci and for *diorismi*; the most and prettiest of theses theorems are new, and it was their discovery which made me aware that Euclid did not work out …

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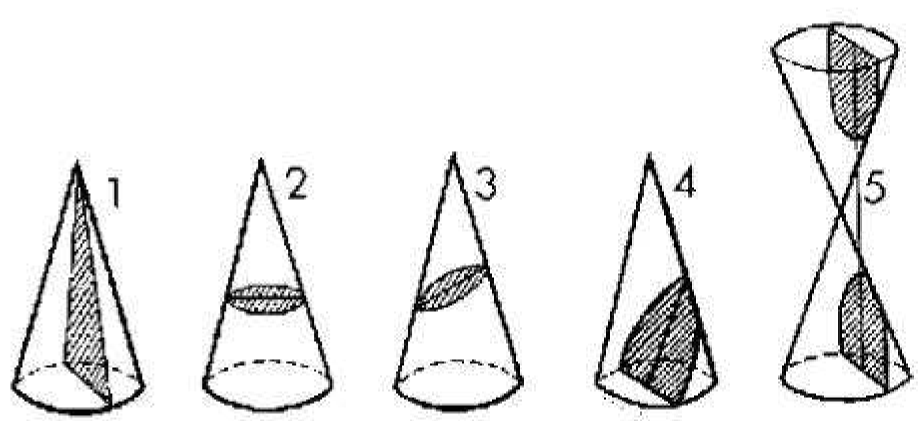
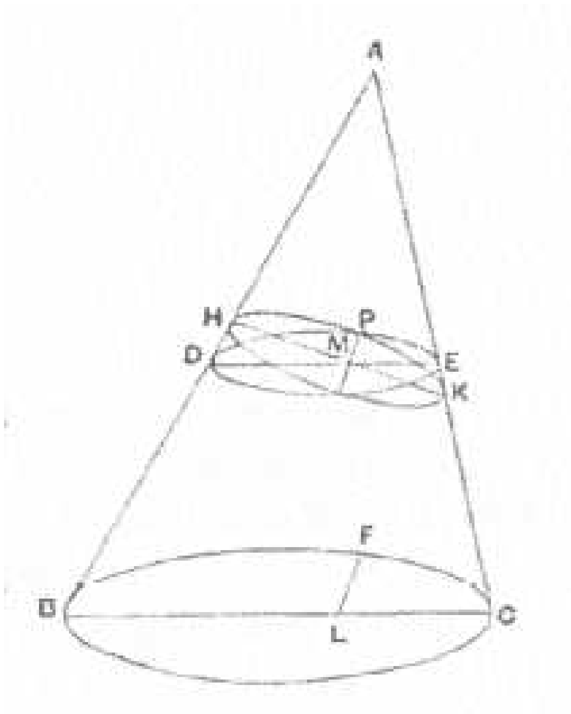
Conics 1

- Conics have a long history that started with Menaechmus (ca 380 - 320 BCE), continued with Euclid and Archimedes, and reached its peak with Apollonius.
- Really complicated because it is 3-D



straight lines circle ellipse parabola hyperbola

An Unusual Circle



- Conic DPE is a circle
- Conic HPK is also a circle if

  < AHK = <BCA

When this is true the section of the cone is called a **subcontrary** section

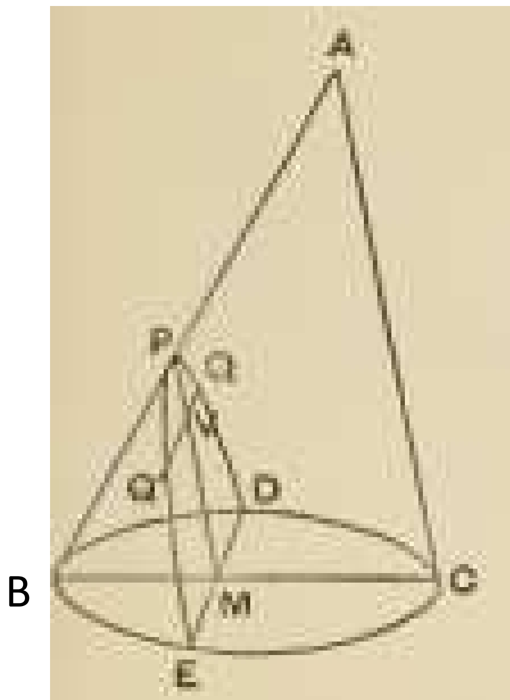
For any point P on the section HPK, it can be shown that

$\underline{HM} * \underline{MK} = \underline{PM}^2$

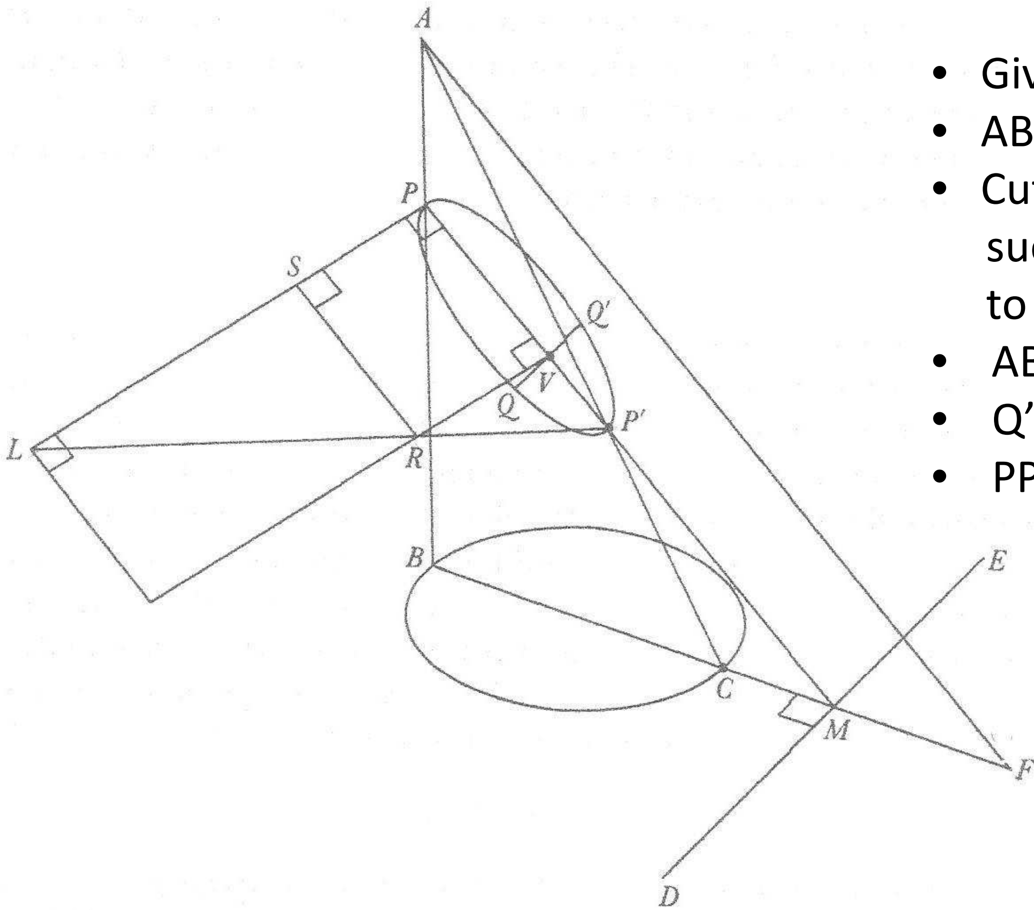
It follows from this that the section HPK is a circle

Except for this special case, no other sections are circles. So what are they?

Definition of a Diameter (Book 1)



If a cone be cut by a plane which intersects the circular base in a straight line (DE) perpendicular to the base of any axial triangle (ABC), the intersection of the cutting plane and the plane of the axial triangle (PM) will be a diameter of the resulting section of the cone.



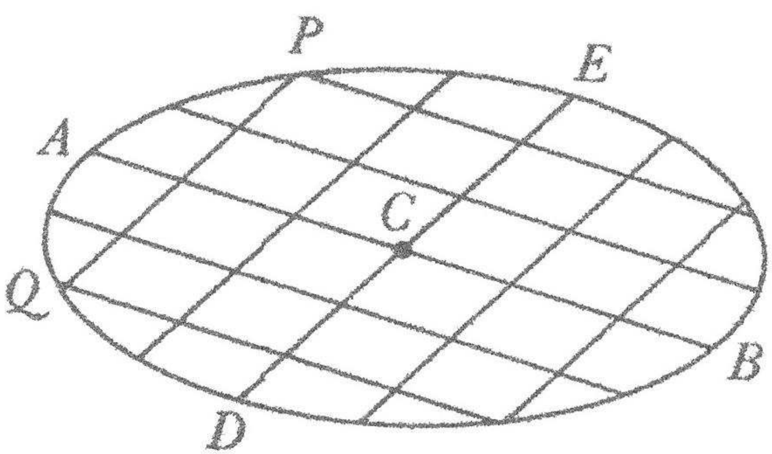
- Given circle BC and vertex A
- ABC is axial triangle
- Cut the cone with plane PDE such that DE is perpendicular to BC extended
- ABC cuts the conic in PP'
- Q'Q is chord parallel to DE
- PP' bisects Q'Q

Apollonius shows that $QV^2 = PV \cdot VR$. QV is our "y". PV is our x. Length PL is 2p and PP' is d. Using the fact that rt LR is similar to rt LP', it can be shown that

$$\mathbf{y^2 = 2px - 2px^2/d} \quad \text{(ellipse)}$$

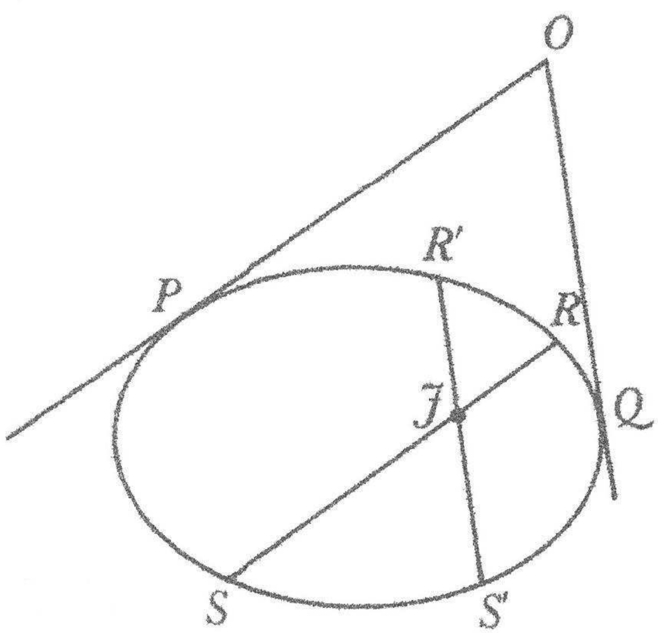
With this construction, d = ∞ for the parabola and if the minus sign is changed to a plus sign an hyperbola is obtained.

Concepts, Book 2



- Many diameters – largest is major axis and smallest is minor axis
- Definition of conjugate diameter
- Construction of tangent

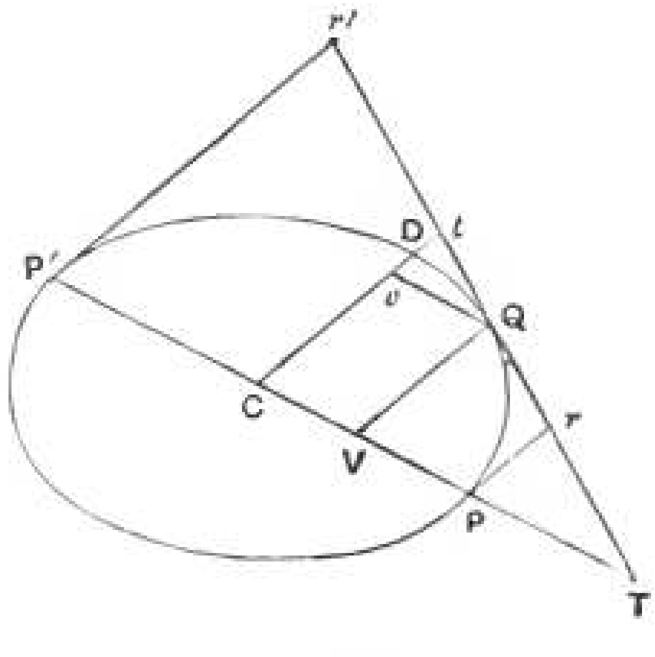
Interesting properties



- OP and OQ are tangents
- RS any chord || OP
- R'S' any chord || OQ
- RS and R'S' intersect in J
- $(RJ \cdot JS)/(R'J \cdot JS') = OP^2/OQ^2$

- This is a generalization of the theorem for a circle.

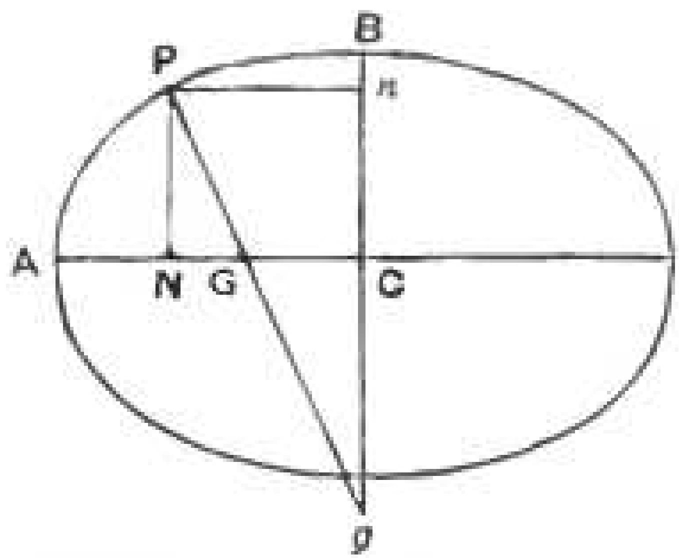
Book III, Proposition 42



- If the tangents at the extremities of a diameter PP' of a central conic be drawn, and any other tangent meet them in r, r' respectively, then

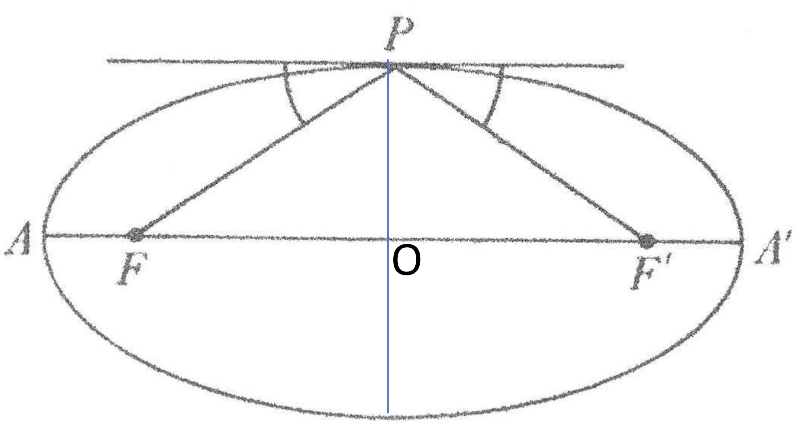
$$Pr \times P'r' = CD^2$$

Book V, Proposition 91



- If g be on the minor axis of a ellipse, and gP is a maximum straight line from g to the curve, and if gP meets the major axis in G, GP is a minimum straight line from G to the curve.
- The concept of minimum and maximum distances is pretty sophisticated.

Foci of Ellipse

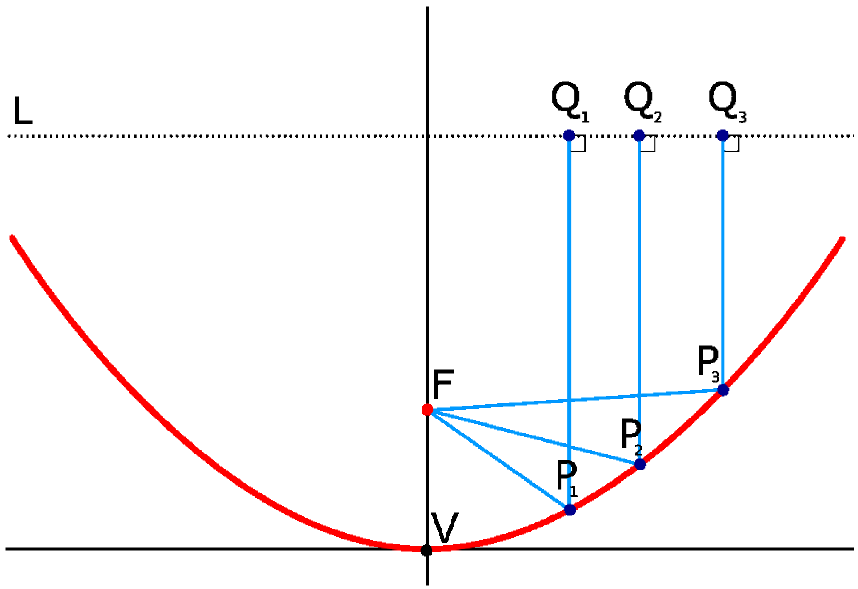


- The foci, points F and F' are determined by PF' = OA' and PO

  FO = OF' = sqrt($AO^2 – PO^2$)
- PF + PF' = 2 AO for P anywhere on the ellipse.
- For P anywhere on the ellipse, the angles between the tangent at P and PF and PF' are equal
- Properties of an ellipsoidal dome

The principle was used in the construction of "whispering galleries" such as in St. Paul's Cathedral in London. If a person whispers near one focus, he can be heard at the other focus, although he cannot be heard at many places in between.

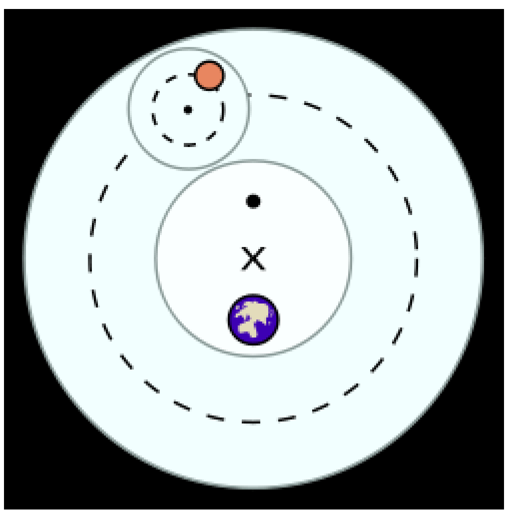
Parabolic Mirror



- Before Apollonius it was believed that a spherical mirror focused light to a point. Apollonius proved that only a parabolic mirror focused light at a point

Apollonius the Astronomy

- Ptolemy :
  - Earth (center), Moon, Mercury, Venus, Sun, Mars, Jupiter, and Saturn.
  - The inequalities in the motions of these heavenly bodies necessitated either a system of deferents and epicycles or one of movable eccentrics (both systems devised by Apollonius of Perga, the Greek geometer of the 3rd century BC) in order to account for their movements in terms of uniform circular motion.



The basic elements of Ptolemaic astronomy, showing a planet on an epicycle (smaller dotted circle) , a deferent (larger dotted circle), and an equant (larger black dot).

http://www.windows2universe.org/mars/mars_orbit.html

Think Kepler (1571 – 1630 CE)

- Kepler’s First Law
  - The orbits of the planets are ellipses, with the Sun at one focus of the ellipse.
- Kepler’s Second Law
  - The line joining the planet to the Sun sweeps out equal areas in equal times as the planet travels around the ellipse.
- Kepler’s Third Law
  - The ratio of the squares of the revolutionary periods for two planets is equal to the ratio of the cubes of their semimajor axes:
- Anyone who ... points the way will be for me the great Apollonius," Kepler wrote. Here Kepler is seeking help in verifying the 2nd law.

Planetary Orbits

Planet	a	e	i
Mercury	0.387	0.206	7° 00'
Venus	0.723	0.007	5° 24'
Earth	1.000	0.017	0° 00'
Mars	1.881	0.093	1° 51'
Jupiter	11.857	0.048	1° 18'
Saturn	29.42	0.056	2° 29'
Uranus	83.75	0.046	7° 00'
Neptune	163.72	0.009	1° 46'
Pluto	248.0	0.249	17° 09

Apollonius the Great Geometer

- Apollonius's contributions to the development of mathematics are countless. Apollonius showed how to construct the circle, which is tangent, to three given circles. He extended Euclid's theory of irrationals and improved Archimedes's approximation of ‘pi.' Apollonius showed that parallel rays of light are not brought to a focus by a spherical mirror and discussed the focal properties of a parabolic mirror. In his mathematical astronomy studies he found the point where a planet appears stationary, namely the points where the forward motion change to a retrograde motion or the converse. To top everything off, Apollonius developed the hemicyclium, a sundial which has the hour lines drawn on the surface of a conic section.
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