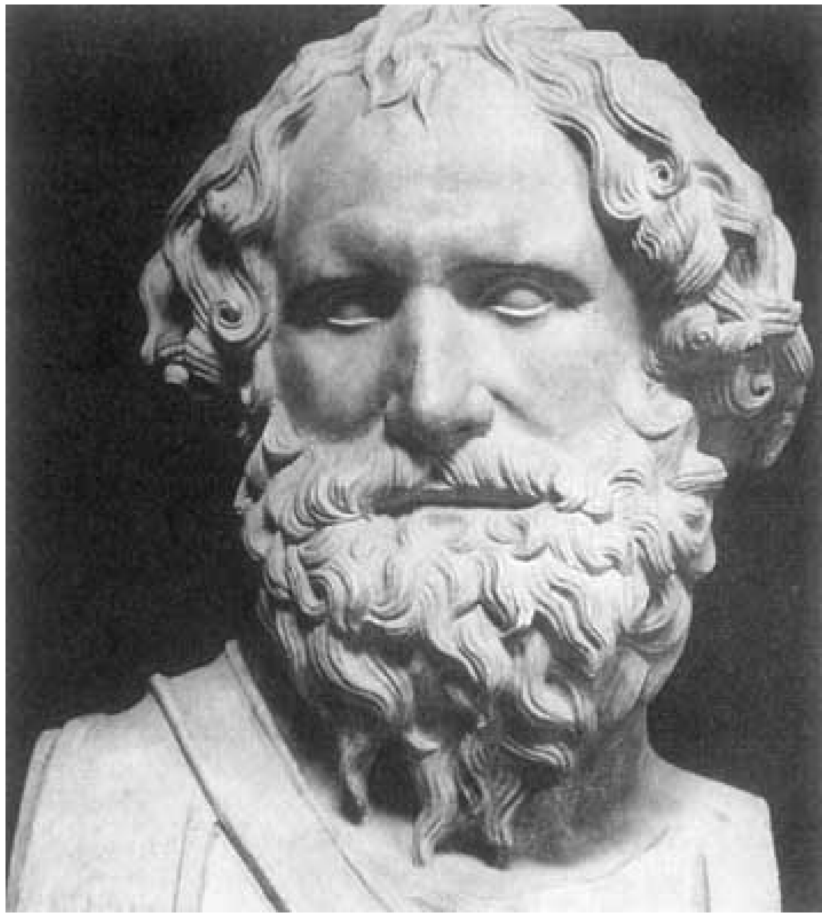
Archimedes - Master of Thought
OLLI Summer 2014

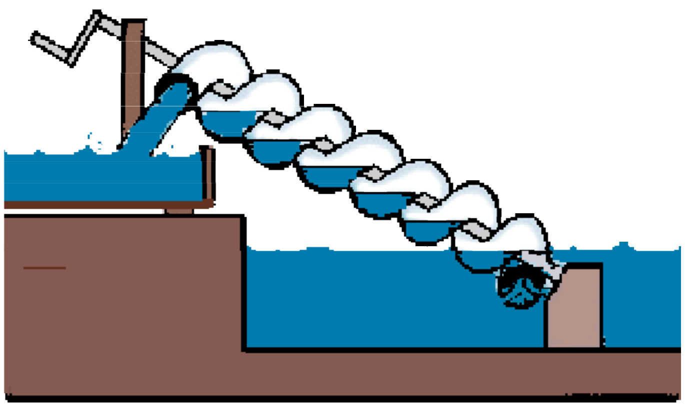


Bio

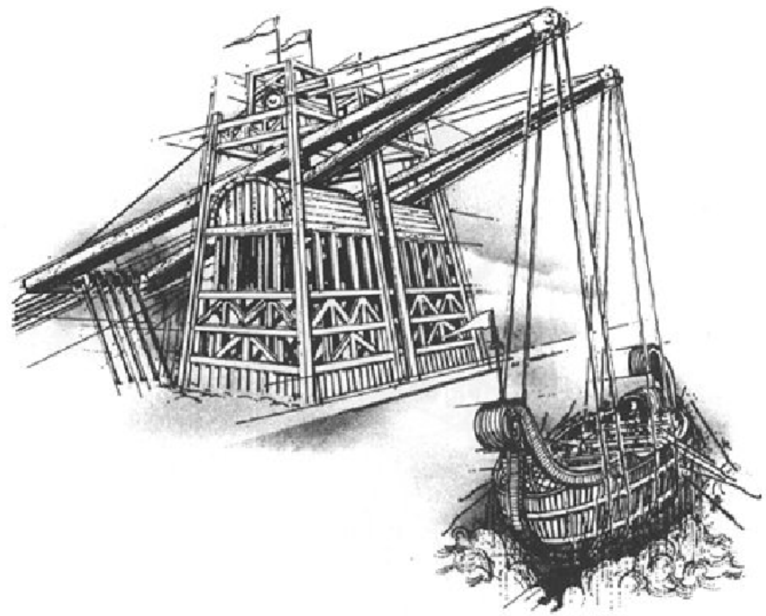
- A life of ***A*** was written by Heracleides, but this biography has not survived, and such particulars as are known have been collected from various sources. Eutocius (ca. 480 – ca. 540 CE) mentions this work in his commentary on ***A***'s *Measurement of the circle* and in other works.
- According to Tzetzes (12th century CE) ***A*** died at the age of 75, and, as he perished in the sack of Syracuse (BCE 212) he was probably born about 287 BCE.
- He was the son of Pheidias the astronomer (mentioned in the Sand-reckoner) and knew and was possibly related to king Hieron and his son Gelon
- He spent time at Alexandria where he probably studied with the successors of Euclid.
- After his return to Syracuse he devoted his life to mathematical research.
- Appears as a historical figure in 216 - 212 BC during the siege and capture of Syracuse

Archimedes and the Burning Mirror

Inventions



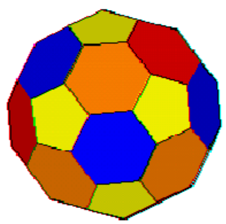
Screw Pump



The Archimedes Claw

Lost Works

- Polyhedra – Contains 5 regular polyhedra and 13 semi-regular polyhedra containing more than one regular polygon



- Book on numbers – Contains material on large numbers similar to what is in the Sand Reckoner
- Book on Balances or Levers
  - Probably where ***A*** proves when an object hangs at rest from one point the CG is directly below the balance point.
- Books on the CG, optics, and the construction of a sphere representing the motions of the sun, moon, and planets around the earth.

Codices A and B

- Leo the Mathematician (ca. 790 – 869 CE)
  - Byzantine philosopher
  - Had Archimedes mathematical treatises copied and bound into a volume (Valla Codex or Codex A)
  - Codex Mechanicorum or Codex B created at about the same time, possibly by Leo
  - Codices A & B are the root sources for almost all of Archimedes work known today. (Heiberg)
  - Codex B disappeared in the 14th century and codex A in the 16th although various Latin and Arabic translations still existed.

Codex C

- Codex C
  - By 1000 CE codex C was created containing parts of A & B and the Method and Stomachion.
- Fourth Crusade of 1204
  - Venetian Crusaders stopped short of the holy land and attacked Constantinople
  - Codices A and B shipped out of Constantinople and Codex C tossed on the recycling heap to be used as a source of parchment
  - In 1229 a scribe “erased” ***A*** text from Codex C and wrote an Eastern Orthodox liturgical guide (The ***A*** Palimpsest)
- Codex C on the move
  - Returns to Constantinople in the 1840s

Mar Saba Monastery

Codex C (continued)

- 19th Century
  - Constantin Tischendorf discovers Palimpsest in 1844 and steals a sample page but does not realize its significance
  - Cambridge purchases the page from Tischendorf's estate in the 1870s where it hides in plain sight.
  - In 1880s Greek Orthodox Church commissioned a catalog of the church's manuscripts. It took 10 years to complete. In the catalog the palimpsest was described:
    - Circa 12th century, palimpsest with unidentified mathematical text.
  - The catalog author did not understand the mathematics, but did include a sample in the catalog.
  - While reviewing the catalog, Hermann Schone reads the description of the palimpsest and sends a copy to Johan Heiberg who was creating a Latin translation of all existing Archimedian treatises.
  - Heiberg recognized the text as from ***A*** but it appeared to be from a new source.

Archimedes Palimpsest

- 20th Century
  - Heiberg finally sees the palimpsest in 1906, the oldest existing record of ***A***'s works.
  - 1907 Heiberg publishes Greek text of the "Method"
  - In 1921 the Palimpsest disappears from public view only to reappear in the 1990s in terrible condition.
  - October 1998, Palimpsest sold at auction to "Mr. B" for $2.2 M
  - Anonymous is likely to be Jeff Bezos of Amazon.com
  - The Palimpsest is taken apart, carefully resurrected, and in 2005 examined using X-ray technology at Stanford
  - Hundreds of corrections made to Heiberg's translation and filled in numerous gaps. Especially true of the "Method".

Codex C

The Archimedes Palimpsest, as it appeared in 1999, (above) closed and (below) open to a proposition from Archimedes' lost treatise, the *Method of Mechanical Theorems*. Only the overlying religious text is visible. (*Photograph by John Dean*)

Folios 16 verso and 17 recto from the Archimedes Palimpsest, photographed in 1906. Here, Euchologion text runs vertically, perpendicular to the Palimpsest's spine; Archimedes' *On Floating Bodies* runs horizontally, parallel to the spine, in two columns. (*The Digital Palimpsest, http://www.archimedespalimpsest.org/*)

Archimedes: Types of Studies

- Those that prove theorems concerning areas and solids bounded by curves and surfaces.
- Works that geometrically analyze problems in statics and hydrostatics
- Miscellaneous works, especially ones that emphasize counting, such as *The Sand Reckoner*

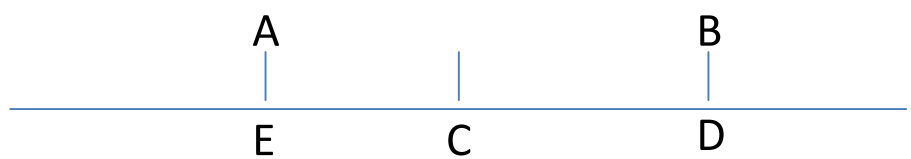
The Works of Archimedes

	A	B	C	Other
On the Equilibrium of Planes	√	√	√	
Quadrature of the Parabola	√	√		
On the Sphere and Cylinder	√		√	
Measurement of a Circle	√		√	
On Spirals	√		√	
On Floating Bodies		√	√	
On Conoids and Spheroids	√			
The Sand-Reckoner	√			
Method of Mechanical Theorems			√	
Stomachion			√	
Book of Lemmas				√
The Cattle-Problem				√

Law of the Balance Bar

On the Equilibrium of Planes, Book 1

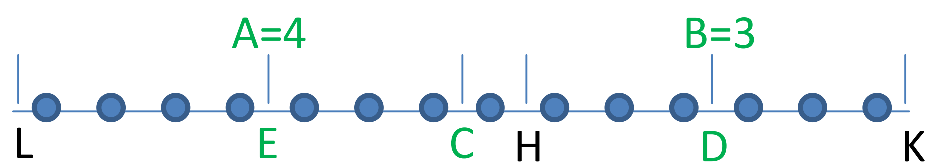
- Proposition 6
  - Given weights A and B, prove the weights are in balance if A x EC = B x CD
  - By Example, A = 4 lb, B = 3 lb, EC = 1 ft, and CD = 4/3 ft



Law of the Balance Bar

On the Equilibrium of Planes, Book 1

- Choose points L, H, and K so that LE = 4/3 ft, CH = 1/3 ft, and DK = 1 ft (Given: A = 4 lb, B = 3 lb, EC = 1 ft, and CD = 4/3 ft)
- Break the 3 and 4 lb weights into ½ lb blocks and place every 1/3 ft along the bar



- The first 8 blocks weigh 4 lbs and are symmetrically positioned around E so they are equivalent to weight A at E. The 6 other blocks are equivalent to weight B at D.
- All the 14 block are symmetrically placed around C so the system is balanced at C.

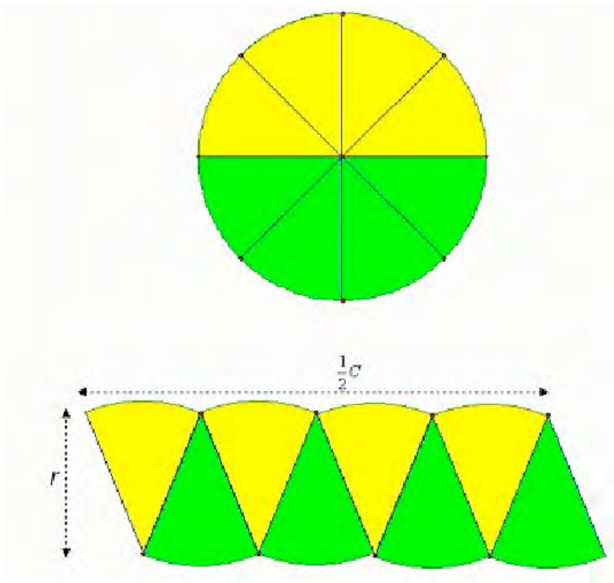
Center of Gravity of a Triangle

On the Equilibrium of Planes, Book 1

- Propositions 6, 7 - Two magnitudes, whether commensurable [Prop 6] or incommensurable [Prop 7], balance at distances reciprocally proportional to the magnitudes.
- Proposition 13 - In any triangle the CG lies on the straight line joining any angle to the middle point of the opposite side.
- Proposition 14 - The CG of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively.
- Did Archimedes invent the concept of the Center of Gravity?
  - It is debatable.

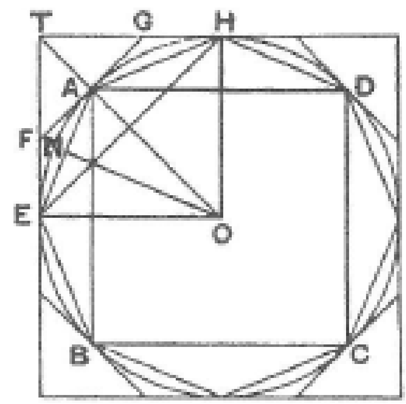
Measurement of a Circle

Proposition 1

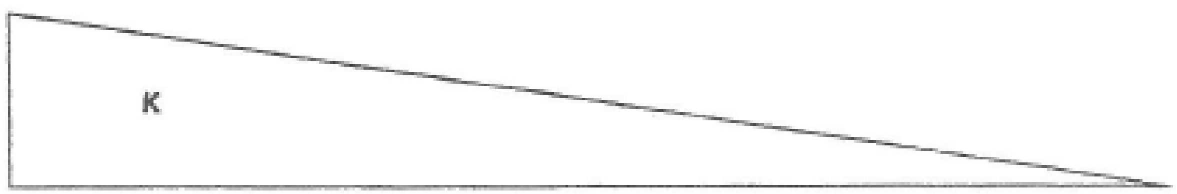


- The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle.

Archimedes Proof



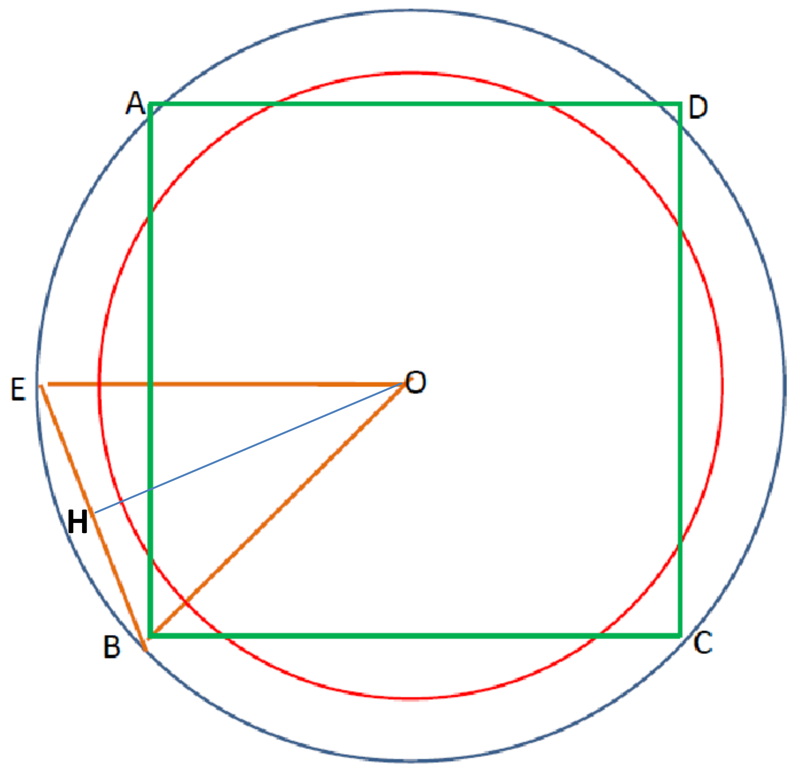
Radius (r)



Circumference (c)

The area of K = c x r/2.   If the area of a circle of radius r is not equal to K, it must be either greater or less. Assume it is greater than K. Then, there is a circle of area K and radius less than r centered on O that can be drawn inside the given circle.

Archimedes Proof -2



- Inscribe a square ABCD inside the circle of radius r, bisect arc AB, then bisect the halves, and so on, until the sides of the inscribed polygon are between the original circle (r) and circle K.
- Consider triangle EBO. Its altitude is less than r and the side EB is shorter than the arc EB. Therefore the area of the polygon is less than K = cr/2.
- Impossible, therefore the area of the circle of radius r cannot be greater than K
- A similar argument is used to show that the area of the circle cannot be less than K

Measurement of a Circle

Propositions 2 and 3

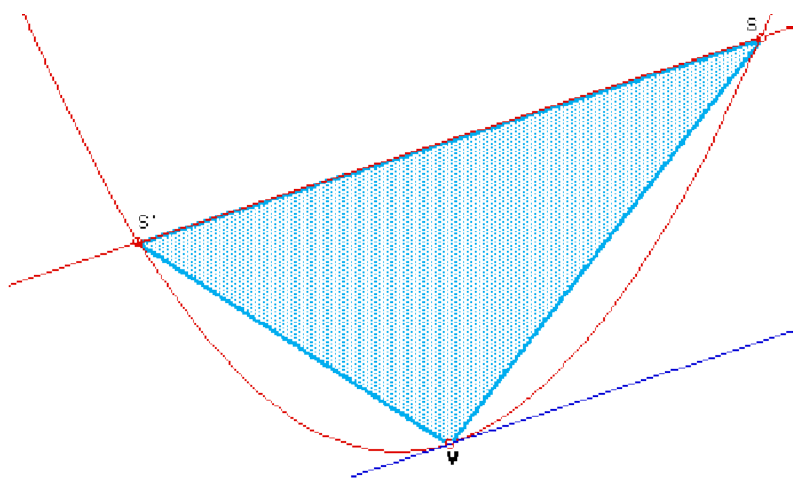
- In proposition 2, ***A*** gives the result that the area of a circle is to the square on its diameter as 11 to 14. This is equivalent to π ≈ 22/7.
- In proposition 3, ***A*** uses the properties of an 96-sided polygon to show that “The ratio of the circumference of any circle to its diameter is less than 3 1/7 but greater than 3 10/71
- This is equivalent to 3.1408 < π < 3.1429

Calculate Pi

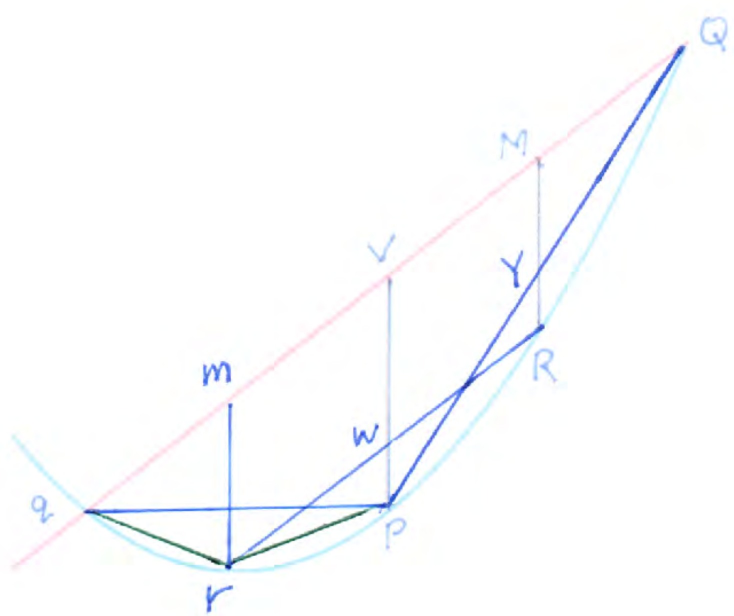
- In proposition 3 Archimedes uses $\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$ without showing how he got it.

Quadrature of the Parabola

- There are 24 propositions in this book.
- Proposition 24 states "Every segment bounded by a parabola and a chord SS' is equal to 4/3 x the triangle which has the same base as the segment and equal height.
- Also addressed in proposition 1 of the Method



Quadrature of the Parabola 2



- V bisects qQ, m bisects qV, and M bisects VQ.
- VP and MR are parallel to axis
- Draw RW parallel to qQ and extend to r.
- ***A*** shows that the area of triangles Prq and PRQ are each 1/8 the area of PqQ
- Prq + PRQ = ¼ PqQ

If the process is repeated starting with parabolic segments Prq and PRQ then the the sum of the areas of the 4 small triangles in the segments is 1/16 PqQ. Continuing the process shows that the area of the original parabolic segment is 4/3 PqQ.

Quadrature of the Parabola 3

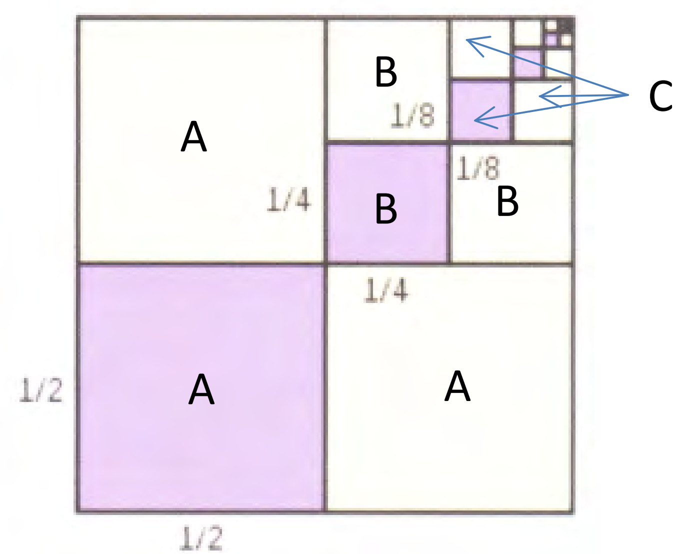
- Summing the series
  - How did ***A*** show that
    - 1 + 1/4 + 1/16 + 1/64 + … = 4/3

3A + 3B + 3C + … = 1

A + B + C + … = 1/3

1/4 + 1/16 + 1/64 + … = 1/3

1 + 1/4 + 1/16 + 1/64 + … = 4/3



Hydrostatics (Book 1)

- P 3: Of solids those which, size for size, are of equal weight with a fluid will, if let down into the fluid, be immersed so that they do not project above the surface but do not sink lower.
- P 4: A solid lighter than a fluid will, if immersed in it, not be completely submerged, but part of it will project above the surface.
- P 5: Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced.
- P 7: A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced.

Eureka

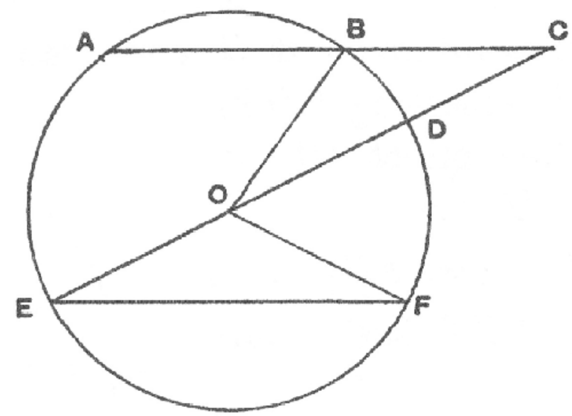
- According to Vitruvius (1st century BCE), King Hieron of Syracuse asked ***A*** to determine if a crown he purchased was gold and to do it without damaging the crown.
- Legend has it that while taking a bath ***A*** observed that the water rose when he stepped into it. When he realized he could measure the volume of the crown by displacement, he jumped out of the tub and ran through Syracuse naked shouting eureka.
- In 1586 Galileo pointed out that using a measurement of the displaced water to determine the volume of the crown could not have been made with sufficient accuracy in 200 BCE to work.
- Galileo suggested another method that could been used.

Trisection of an Angle

The Book of Lemmas

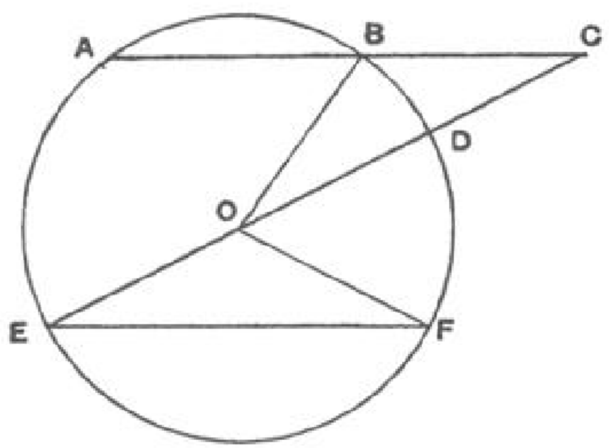
Proposition 8.

If AB be any chord of a circle whose centre is O, and if AB be produced to C so that BC is equal to the radius; if further CO meet the circle in D and be produced to meet the circle a second time in E, the arc AE will be equal to three times the arc BD.



Draw the chord EF parallel to AB, and join OB, OF.

Trisection of an Angle



- BC = BO implies
- Triangle BOC is isosceles
- Draw EF parallel to AC
- < BOC = < BCO = α
- < BCO = < OEF = < OFE = α
- < DOF = 2α
- < BOF = α + 2α = 3α
- arc AE = arc BDF implies
  - arc BD = 1/3 arc AE

AE is arc to be trisected. Draw EO, extend to D and beyond. Take a straight edge and use the compass to mark two points on it for the radius of the circle. Place the straight edge through point A so that one of the points is on line ED extended (C) and the other is on the circle (B). Arc BD is 1/3 arc AE.

The Method and Infinity

- Heiberg published his translation of Codex C in 1907. Because of the binding of the Palimpsest and the hidden nature of the text, parts of Codex C were not visible.
- It was not until after 2001 and the work of William Noel and his team at the Walters Art Museum in Baltimore that the importance of this high tech reading of Codex C became apparent. Some of the blanks of Heiberg's translation could now be read.
- My comments that follow are from Reviel Netz and William Noel's book "The Archimedes Codex – How a Medieval Prayer Book Is Revealing The True Genius Of Antiquity's Greatest Scientist"

The Method and Infinity

- The Greeks invented mathematics as a precise, rigorous science. They avoided paradox and mistakes. In doing so, they avoided the pitfall of infinity. Their science was based on numbers that can be as big as you wish, or as small as you wish, but never *infinitely* big or small. Numbers that are as big or small as you wish are known as “potentially infinite”, instead of actually infinite. The Greeks did not use actual infinity.
- Galileo and Newton incorporated new techniques into mathematics by employing actual infinity, but there was a price to pay. Paradoxes and errors followed.
- In the 19^{th} century, mathematicians created new techniques for dealing with infinity.

The Method and Infinity

- Consider the two sets of numbers:

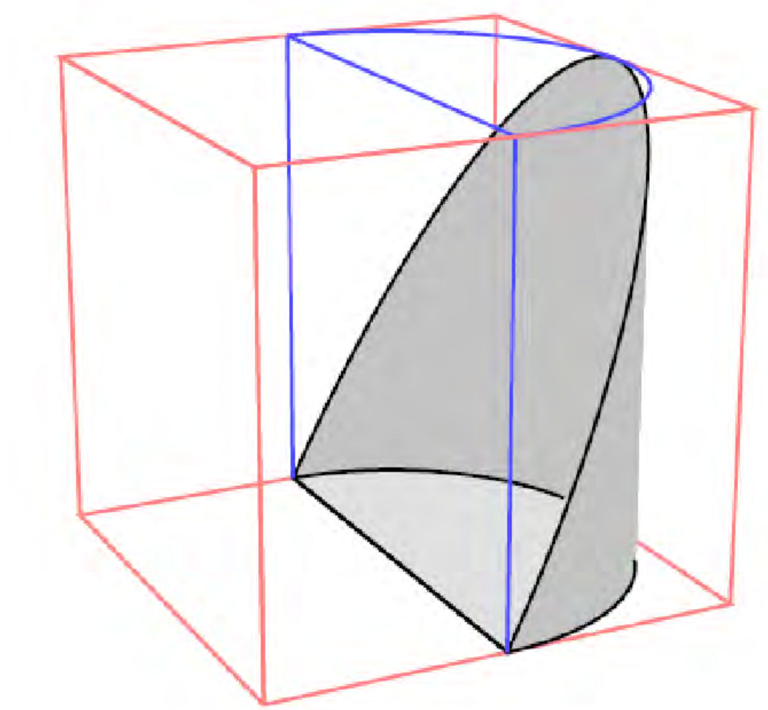
1	2	3	4	5	…
2	4	6	8	10	…

- The bottom row does not get exhausted. For each whole number there is an even number and vice versa. The number of whole numbers is the same as the number of even numbers even though, in some sense, there are twice as many whole numbers.
- In infinity "normal" concepts collapse: a collection may be equal to its half.
- We cannot count on ordinary rules of addition and summation.
- Even in the *Method* – so Netz and Noel thought back in January 2001 – Archimedes did not treat actually infinity.

The Method and Infinity

- In propositions 1 – 13 in the *Method,* Archimedes procedes in typical Greek fashion and uses a combination of physics, mathematics, and potential infinity.
- Proposition 14 is different. It is neither an orthodox proof, nor is it like the first 13 propositions of the *Method*. It does not rely on the combination of the application of physics to mathematics and infinite summation. Instead, it is based on infinite summation alone.
- Heiberg was only able to read the beginning and end of P14.

The Method and Infinity



http://www.calstatela.edu/faculty/hmendel/Ancient%20Mathematics/Archimedes/Archimedes%20Method/Prop14/Arch.Method.Prop.14.html

The Sand Reckoner

- The Problem
  - How many grains of sand would it take to fill the universe?
- The issues
  - What model of the universe to use.
  - How big is the universe?
  - How to expand Greek arithmetic to handle very large numbers?
- When and why was the Sand Reckoner written?
  - 216 BCE a few years before Archimedes' death
  - For King Gelon II and the non mathematician shortly before Gelon's death
  - To introduce a method for writing very large numbers.

The Sand Reckoner 2

- Model of universe
  - Heliocentric model proposed by the astronomer-mathematician Aristarchus of Samos. Although there is some debate, most accept Aristarchus as the first to propose it.
  - Aristarchus wrote:
    - The distance of the stars bears the same relation to the diameter of the Earth's orbit as the surface of a sphere bears to its center.
  - ***A*** interpreted Aristarchus statement to mean:
    - The distance of the stars bears the same relation to the diameter of the Earth's orbit as the diameter of the Earth's orbit bears to the diameter of the Earth.
  - No one knows why ***A*** chose this model of the universe and not the Earth centered model, but it does give a much larger answer.

The Sand Reckoner 3

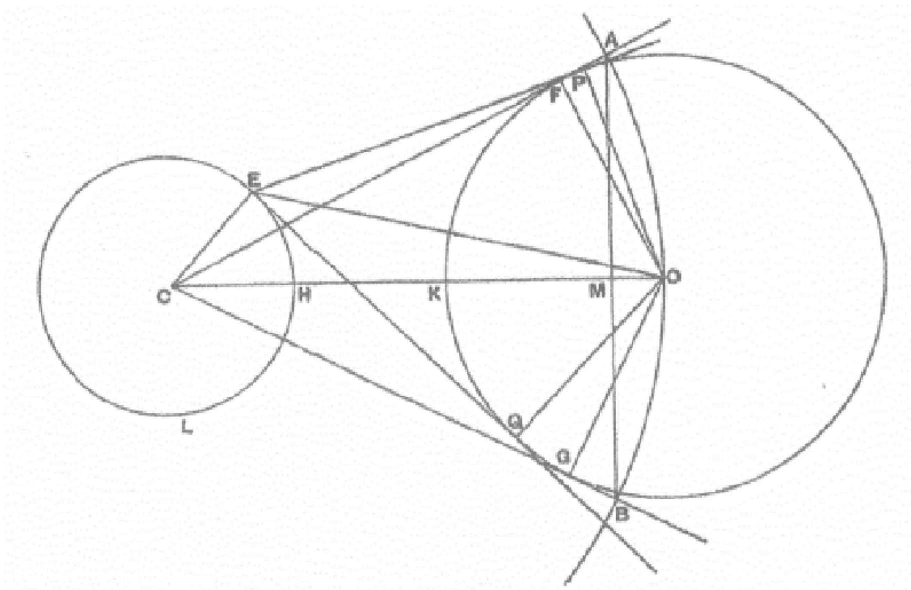
- In 216 BCE, Earth's circumference estimated to be 300,000 stadia
  - The length of a stadia is 600 ft +/- 85ft depending on the city.
  - Using 600 ft yields a diameter of 10850 land miles (today – 7926 land miles)
  - Later Eratosthenes improved on the estimate to about 252,000 stadia. What stadia he used is unknown, but he made all his measurements in Egypt where the stadia was about 157.5 m.
    - Corresponds to earth's diameter = 7850 miles (actual about 7920 miles)
  - ***A*** was going for big numbers and measurements are fraught with errors so he assumed the earth's circumference could be as large as 3,000,000 stadia.
  - For this exercise ***A*** overestimated everything

The Sand Reckoner 4

Archimedes' Assumptions

- (1) The circumference of the earth is about 3,000,000 stadia
- (2) The $D_{sun >} D_{earth} > D_{moon}$
- (3) $D_{sun} \sim 30\ D_{moon}$
- (4) D_{sun} is greater than the side of the chiliagon inscribed in the circular orbit of the earth around the sun.

The Sand Reckoner 5



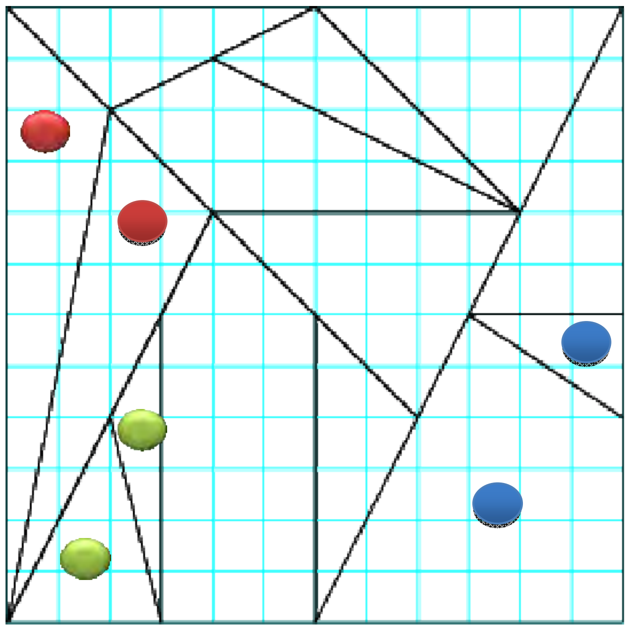
- E is the location of the "eye"
- Line EQ is tangent to both the earth and the sun and provides the position of the sun when it is rising and one just sees the whole sun.
- 90°/164 > angle PEQ >90°/200
- ***A*** corrects for parallax
  - Angle ACB > 90°/203
- Side of chiliagon subtends angle of 360°/1000 = 90°/250
- D_{sun} > side of chiliagon
- Issue of refraction

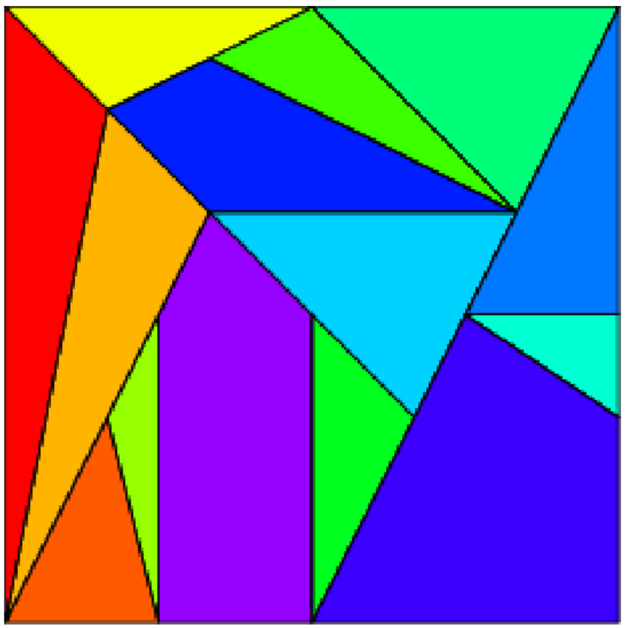
The Sand Reckoner 6

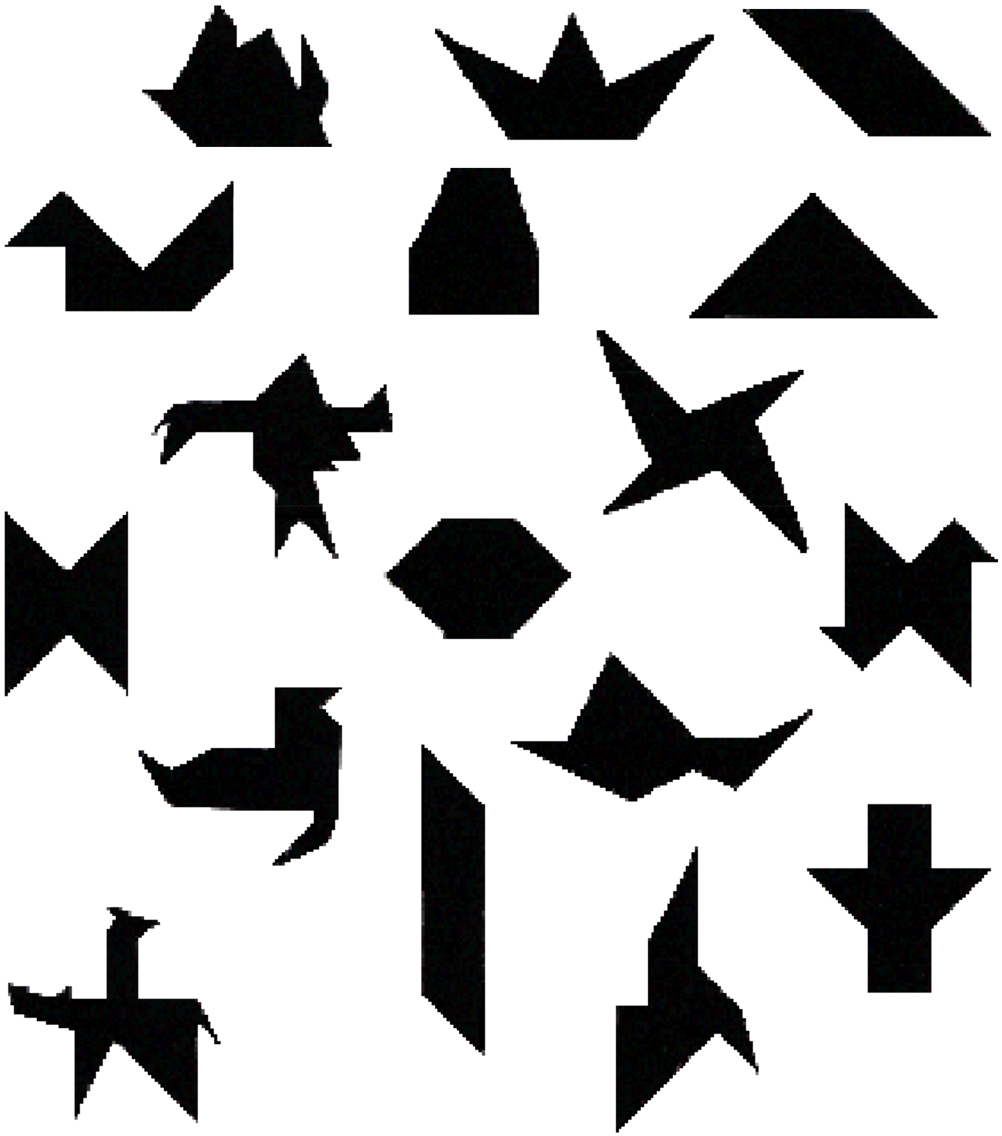
- D_{sun} > side of chiliagon 1
- 1000 D_{sun} > perimeter of chiliagon > 3 $D_{earth\ orbit}$ 2
- The $D_{sun\ >}$ D_{earth} > D_{moon} (assumption 2) 3
- D_{sun} ~ 30 D_{moon} (assumption 3) 4
- 30 D_{earth} > 30 D_{moon} ~ D_{sun} 5
- 30,000 D_{earth} > 1000 D_{sun} from 5 6
- 30,000 D_{earth} > 3 $D_{earth\ orbit}$ from 6 and 2
- 10,000 D_{earth} > $D_{earth\ orbit}$
- Aristarchus: $D_{universe}/D_{earth\ orbit} = D_{earth\ orbit}/D_{earth}$
- $D_{universe} = D_{earth\ orbit}\,(D_{earth\ orbit}/D_{earth}) < 10{,}000\ D_{earth\ orbit}$
- $D_{universe} < 10^8\ D_{earth}$

- This calculation is followed by an estimate of the number of grains of sand in the universe.

Stomachion







Classic Shapes

The Cattle Problem

- It is required to find the number of bulls and cows of each of the four colors, or to find 8 unknown quantities. The first part of the problem connects the unknowns by seven simple equations; and the second part adds two more conditions to which the unknowns must be subject.
- Let W, w be the number of white bulls and cows respectively. Similarly B, b for black, Y, y for yellow, and D, d for dappled.

$$W = (1/2 + 1/3)B + Y, \quad B = (1/4 + 1/5)D + Y$$

$$D = (1/6 + 1/7)W + Y$$

$$w = (1/3 + 1/4)(B + b), \quad b = (1/4 + 1/5)(D + d)$$

$$d = (1/5 + 1/6)(Y + y), \quad y = (1/6 + 1/7)(W + w)$$

The Cattle Problem 2

When the white bulls joined in number with the black, they stood firm with depth and breadth of equal measurement.

W + B = a square (simple version)

W + B = a product of two whole numbers (hard version)

Y + D = a triangular number

In the solution to the hard problem each of the eight numbers has about 206,545 digits. (Ilan Vardi)

Archimedes Mathematical Achievements

Area of a circle (area of right Δ with base = circumference and altitude = radius)

- Circumference = $2\pi r$, $3\frac{10}{71} < \pi < 3\frac{1}{7}$
- Area of a segment of a parabola
- Area of an ellipse
- Volume and surface area of a sphere
- Volumes of various "solids of revolution" obtained by rotating a curve about a fixed straight line.
- Law of the lever , center of gravity, hydrostatics
- How to work with very large numbers

Archimedes Mathematical Achievements (cont.)

- One of the methods he used to find the areas, volumes and surface areas of many bodies was an early form of integration. This was considered his greatest mathematical invention, leading to the field of Calculus. To determine the area of sections bounded by geometric figures such as parabolas and ellipses, Archimedes broke the sections into an "infinite" number of triangles and added the areas together.

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