

Euclid of Alexandria

The Writer of the Elements

OLLI

Summer 2014

BIO

- Little is known of Euclid's life. According to Proclus (410-485 A.D.) in his *Commentary on the First Book of Euclid's Elements*, he came after the first pupils of Plato and lived during the reign of Ptolemy I (306-283 B.C.). Pappus of Alexandria (fl. c. 320 A.D.) in his *Collection* states that Apollonius of Perga (262-190 B.C.) studied for a long while in that city under the pupils of Euclid. Thus it is generally accepted that Euclid flourished at Alexandria in around 300 B.C. and established a mathematical school there.

Ptolemy I (323 – 285 BC)

- Upon the death of Alexander the Great in 323 BC, the throne of Egypt fell to Ptolemy one of Alexander's trusted commanders
- It is said that Ptolemy once asked Euclid if there was in geometry any shorter way than that of the *Elements*. Euclid replied that there was no royal road to geometry.
- From the time of Archimedes onwards, the Greeks referred to Euclid as the writer of the *Elements* instead of using his name.

Euclid's Elements (300 BC)

The greatest mathematical textbook of all time

- Book I. The fundamentals of geometry: theories of triangles, parallels, and area.
- Book II. Geometric algebra.
- Book III. Theory of circles.
- Book IV. Constructions for inscribed and circumscribed figures.
- Book V. Theory of abstract proportions.
- Book VI. Similar figures and proportions in geometry.
- Book VII. Fundamentals of number theory.
- Book VIII. Continued proportions in number theory.
- Book IX. Number theory.
- Book X. Classification of incommensurables.
- Book XI. Solid geometry.
- Book XII. Measurement of figures.
- Book XIII. Regular solids.

<http://aleph0.clarku.edu/~djoyce/java/elements/>

Euclid's Other Surviving Works

- Data, Phenomena, Optics, and On Division of Figures
 - Data
 - A collection of 95 exercises for students
 - Defines and discusses what must be “given” in a geometrical proof.
 - Phenomena
 - The study of spherical geometry for the purpose of explaining planetary motions
 - Optics
 - Earliest surviving Greek treatise on perspective
 - Euclid relates the apparent size of an object to its distance from the eye and investigates the apparent shapes of cylinders and cones when viewed from different angles.
 - On Division of Figures
 - Survives only in Arabic translation
 - Consists of 36 propositions concerning the division of various figures into two or more equal parts or parts in given ratios.

Euclid's Lost Works

- Conics, Porisms, Pseudaria, and Surface Loci
 - Conics
 - Euclid's *Conics* predated by a half-century the famous work by Apollonius on the same subject. However, Euclid's treatment was most likely a compilation of previously known information.
 - Pseudaria
 - Book of Fallacies to warn beginners against different types of fallacies to which they might be susceptible in geometrical reasoning.

Euclid's Approach in the Elements

- Definitions
- Small number of assumptions (postulates)
- Common notions (axioms)
- Propositions (theorems)
 - Hypothesis, Conclusion, and Proof

Book I- Definitions

- Some of these indicate little more than certain concepts will be discussed, such as Def.I.1, Def.I.2, and Def.I.5, which introduce the terms point, line, and surface.
 - 1. A point is that which has no part.
 - 2. A line is breadthless length.
 - 3. The extremities of a line are points
 - 4. A straight line is a line which lies evenly with the points on itself.
 - 5. A surface is that which has length and breadth only.
 - 6. The extremities of a surface are lines.
 - 7. A plane surface is a surface which lies evenly with the straight lines on itself.

Book I- Definitions

- Others are substantial definitions which actually describe new concepts in terms of old ones. For example, Def.I.10 defines a *right angle* as one of two equal adjacent angles made when one straight line meets another.
 - 10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called perpendicular to that on which it stands.
 - 23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.
 - There are 23 definitions.

Book I - Postulates

- Following the list of definitions is a list of postulates. Each of which is accepted without proof.
- Let the following be postulated:
 - 1. It is possible to draw a straight line from any point to any other point.
 - 2. It is possible to produce a finite straight line continuously in a straight line.
 - 3. It is possible to describe a circle with any center and distance.
 - 4. That all right angles are equal to each other.
- Postulates 1 & 2 imply straight lines are unique, two straight lines cannot share a common segment, two straight lines cannot enclose a space.

Book I – Postulate 5

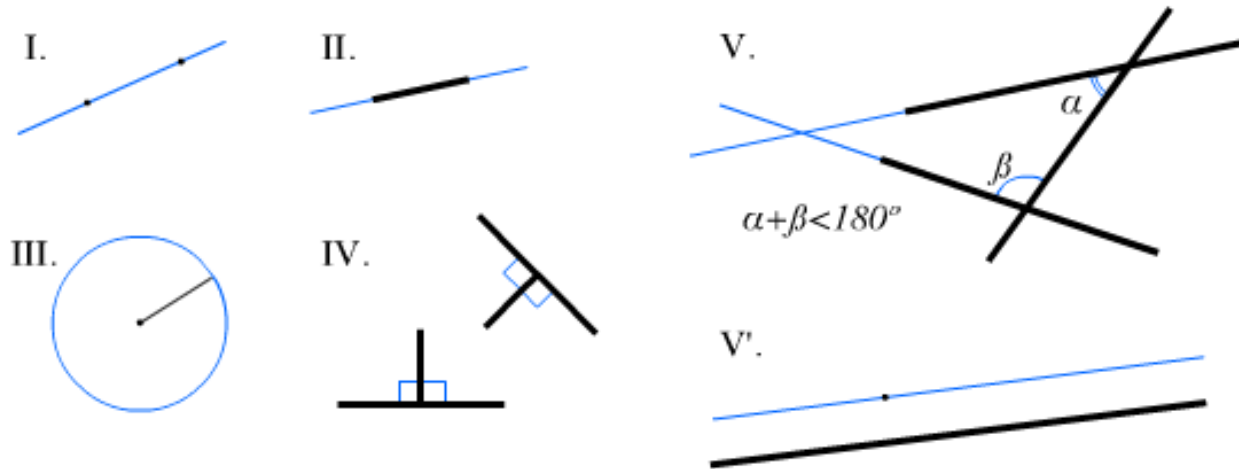
- 5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on the side on which are the angles less than two right angles.

Euclidean geometry is the study of geometry that satisfies all of Euclid's axioms, *including* the parallel postulate. In 1823, Janos Bolyai and Lobachevsky independently realized that entirely self-consistent "non-Euclidean geometries" could be created in which the parallel postulate *did not hold*.

Geometry that is *independent* of Euclid's fifth postulate (i.e., only assumes the first four postulates) is known as absolute geometry (or, in other places known as neutral geometry).

Euclid's Postulates

1. To draw a straight line from any point to any other point
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to each other.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on the side on which are the angles less than two right angles.



Book I - Magnitudes and the Common Notions

- The Common Notions are also axioms, but they refer to magnitudes of various kinds. The kind of magnitude that appears most frequently is that of straight line. Other important kinds are rectilinear angles and areas (plane figures). Later books include other kinds.
 - 1. Things which are equal to the same thing are also equal to one another.
 - 2. If equals be added to equals, the wholes are equal.
 - 3. If equals be subtracted from equals, the remainders are equal.
 - 4. Things which coincide with one another are equal to one another.
 - 5. The whole is greater than the part.

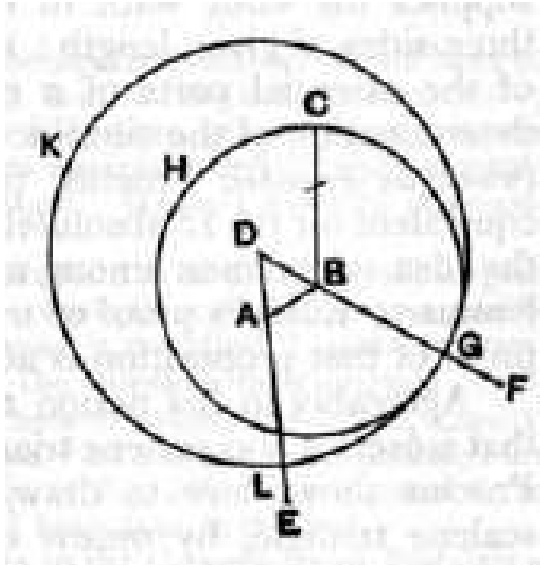
Magnitudes and the Common Notions

- Book III, parts of circumferences of circles, that is, arcs, appear as magnitudes. Only arcs of equal circles can be compared or added, so arcs of equal circles comprise a kind of magnitude, while arcs of unequal circles are magnitudes of different kinds. These kinds are all different from straight lines. Whereas areas of figures are comparable, different kinds of curves are not.

Book I - Propositions

- Following the definitions, postulates, and common notions, there are 48 propositions. Each of these propositions includes a statement followed by a proof of the statement. Each statement of the proof is logically justified by a definition, postulate, common notion, or an earlier proposition that has already been proven.
- P1. On a given finite straight line construct an equilateral triangle
- P2. To place at a given point (as an extremity) a straight line equal to a given line.
- P47. In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Proposition 2



P2. To place at a given point 'A' (as an extremity) a straight line equal to a given line 'BC'.

1) Draw line AB

2) Construct equilateral triangle DAB

3) Extend lines DA & DB to E and F.

4) With B as center and BC the radius, draw circle CGH

5) With D as center and DG as the radius, draw circle GKL

6) $BC = BG$ and $DL = DG$

7) Since $DA = DB$, $AL = BG$

8) Since $BC = BG$, $AL = BC$ (things equal to the same thing are equal to each other).

Other Interesting Propositions in Book 1

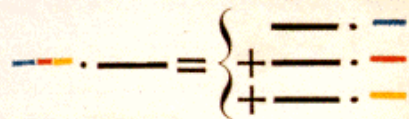
- P5 The base angles of an isosceles Δ are equal.
- P16 An exterior angle of a Δ is greater than either remote interior angle.
- P20 The sum of any two sides of a Δ is greater than the third side. This is close to the shortest distance between two points is a straight line.
- P44 To a given straight line to apply, in a given rectilinear angle, a parallelogram equal to a given Δ .
- P47 The Pythagorean Theorem
- P48 Converse of the Pythagorean Theorem

Euclid Book 2 – Geometric Algebra

- Book 2 contains 14 propositions
- Have discussed that the Greeks did not recognize the existence of irrational numbers so could not handle all lengths, areas etc. numerically. In Book 2 all quantities are represented geometrically, and thereby the problem of assigning numerical values is avoided.
- Numbers are replaced by line segments.
- The product of two numbers becomes the area of a rectangle.
- The product of three numbers becomes a volume



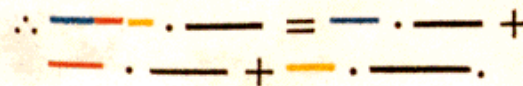
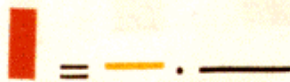
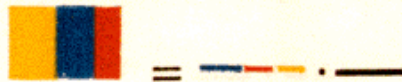
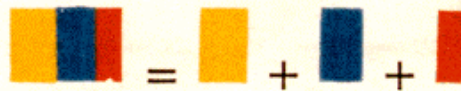
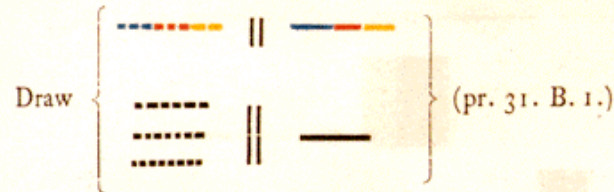
THE *rectangle* contained by two straight lines, one of which is divided into any number of parts,



is equal to the sum of the rectangles contained by the undivided line, and the several parts of the divided line.



Draw \perp and \equiv (prs. 2. 3. B.1.); complete the parallelograms, that is to say,



Q. E. D.

Euclid Book 2 – Geometric Algebra

$$a(b + c + d + \dots) = (ab + ac + ad + \dots)$$

$$(a + b)a + (a + b)b = (a + b)^2$$

$$(a + b)a = ab + a^2$$

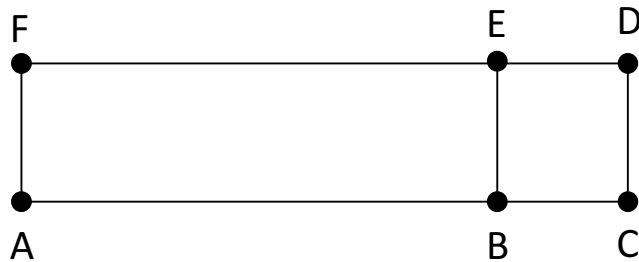
$$(a + b)^2 = a^2 + 2ab + b^2$$

$$ab + \left\{ \frac{1}{2}(a + b) - b \right\}^2 = \left\{ \frac{1}{2}(a + b) \right\}^2$$

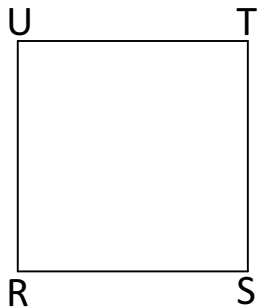
$$(2a + b)b + a^2 = (a + b)^2$$

Euclid Book 2 (What we would do)

- Example: Given a line segment AB and square RT, find BC so that the area of the rectangle AD equals the area of square RT given that $CD = BC$.



$$AB = a, BC = x, CD = x, RS = s$$
$$x(a + x) = s^2, \text{ Solve for } x.$$



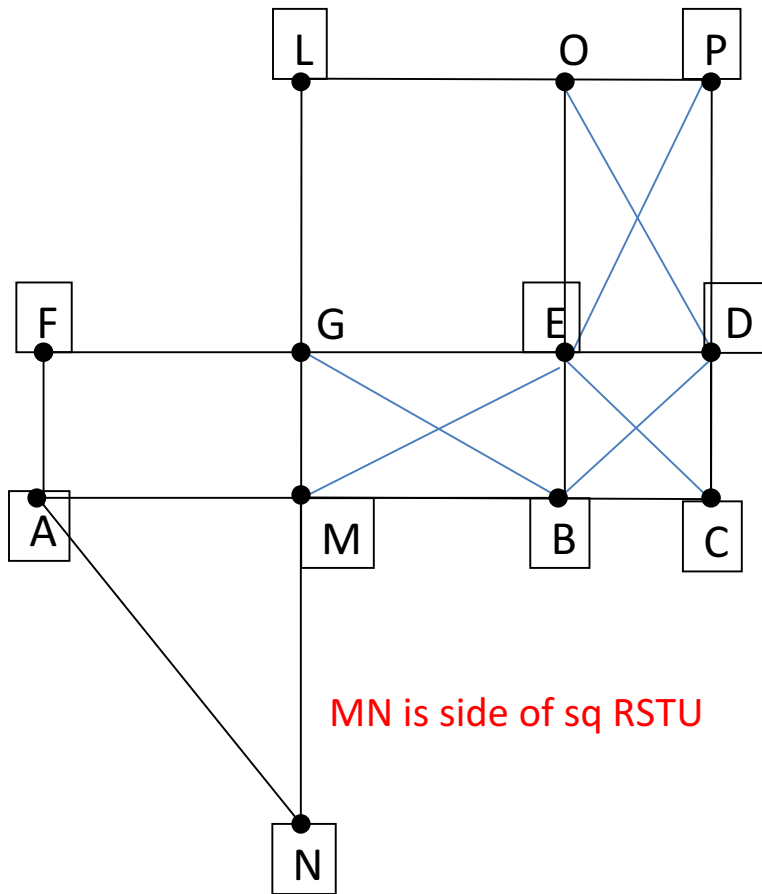
Example: $a = 6$, area of RT = 16.

$$x^2 + 6x - 16 = 0$$

$$(x + 8)(x - 2) = 0; x = 2, x = -8$$

Book 2 Geometric Algebra

What the Greek's Did



- AB is original line segment 1
- M is midpoint of AB, MN is side of square 2
- Construct MC = AN and sq MP of side AN 3
- BC is the solution : Area of rt AD = MN^2 4
- Proof: $AM^2 + MN^2 = AN^2 = \text{sq MP}$ 5
- $\text{sq MP} = \text{rt ME} + \text{sq GO} + \text{rt EP} + \text{sq BD}$ 6
- $AM^2 = \text{sq GO},$ 7
- $\text{sq GO} + MN^2 = \text{rt ME} + \text{sq GO} + \text{rt EP} + \text{sq BD}$ 8
- $MN^2 = \text{rt ME} + \text{rt EP} + \text{sq BD}$ 9
- $\text{rt EP} = \text{rt AG}$ 10
- $MN^2 = \text{rt ME} + \text{rt AG} + \text{sq BD} = \text{rt AD}$ 11

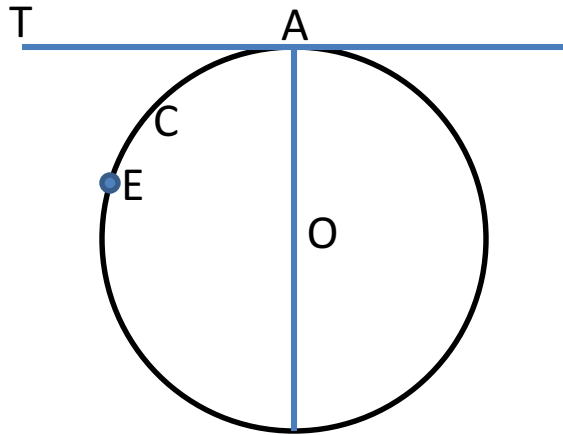
Book 2 Geometric Algebra

- Propositions 12 and 13 are of interest because they foreshadow trigonometry.
- Proposition 12/13 is essentially the law of cosines for obtuse/acute angles for plane triangles
- P 13 In acute-angled triangles the square on the side opposite the acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.

Book 3

- It is generally supposed that the contents of Books 1 & 2 of the Elements are the works of the Pythagoreans
- Books 3 & 4 deal with the geometry of the circle and the material is presumed to have been drawn largely from Hippocrates of Chios.
- The two books are similar to the theorems on circles in today's textbooks.
- Book 3 contains 37 propositions, begins with some definitions appropriate to the geometry of circles and then proceeds to discuss properties of chords, tangents, secants, central and inscribed angles, and so on.

Book 3 Proposition 16

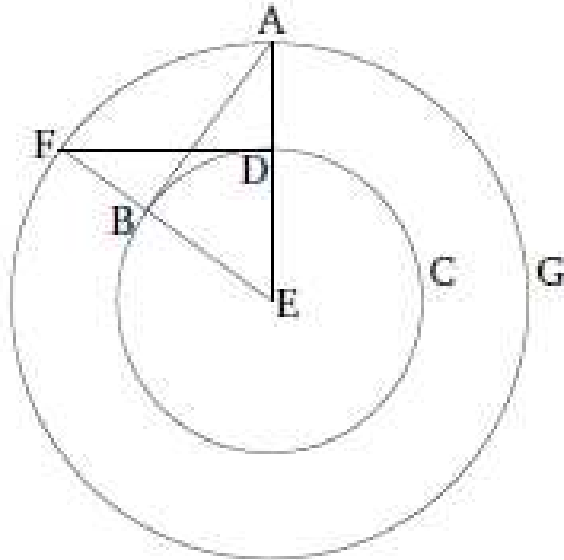


- a) The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle,
- b) and into the space between the straight line and the circumference another straight line cannot be interposed;
- c) further the angle of the semicircle (between the

diameter and the circumference) is greater than any acute angle, and the remaining angle (formed by the semicircle and the tangent line) is less.

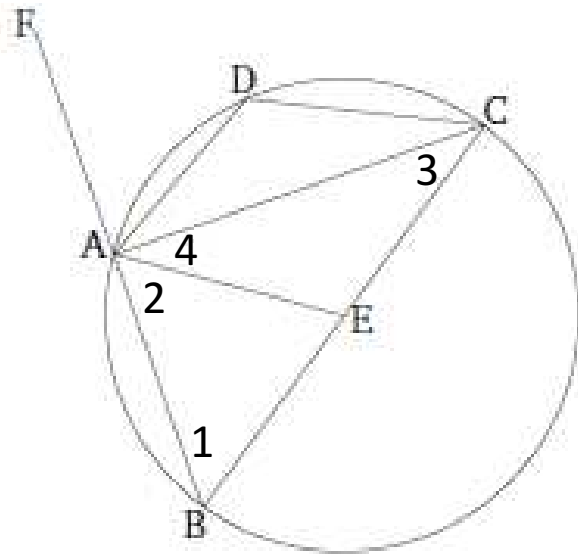
Euclid considers the angle between TA and the arc ACE. Whether this angle, called the hornlike angle, had a definite magnitude was a subject of controversy. P16 says it is smaller than any acute angle formed by straight lines but does not say it is zero.

Book 3, Proposition 17



- From a given point (A) to draw a straight line touching a given circle (BDC).
- Take the center E of the circle, and join AE. Describe the circle FAG with center E and radius EA. Draw DF from D at right angles to EA. Join EF and AB
- The construction is then followed by a proof that it is correct.

Book III, Proposition 31 (Thales)

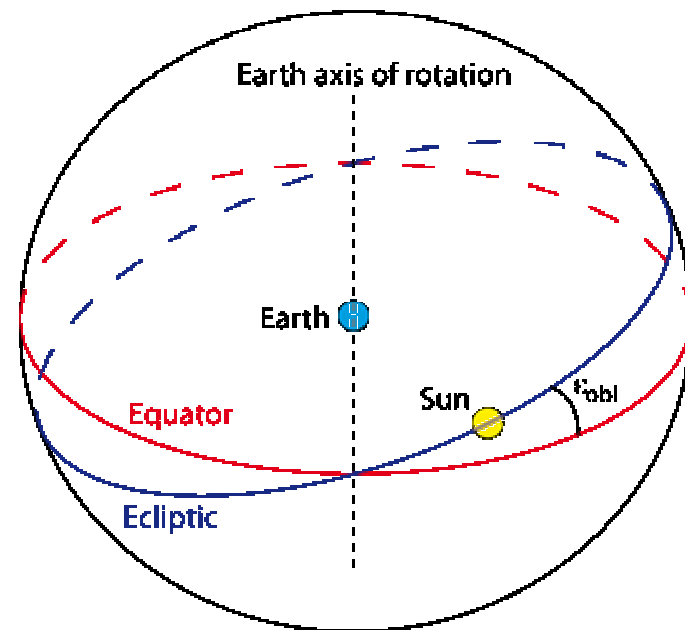


- In a circle the angle in the semicircle is right, ...
- $\angle 1 = \angle 2$
- $\angle 3 = \angle 4$
- $\angle FAC = \angle 1 + \angle 3 = \angle 2 + \angle 4$
- Therefore $\angle FAC = \angle CAB$
- $\angle FAC + \angle CAB = \text{straight angle}$
- $\angle CAB$ is a right angle

- Thales of Miletus (624 – 547 BC)

Book 4

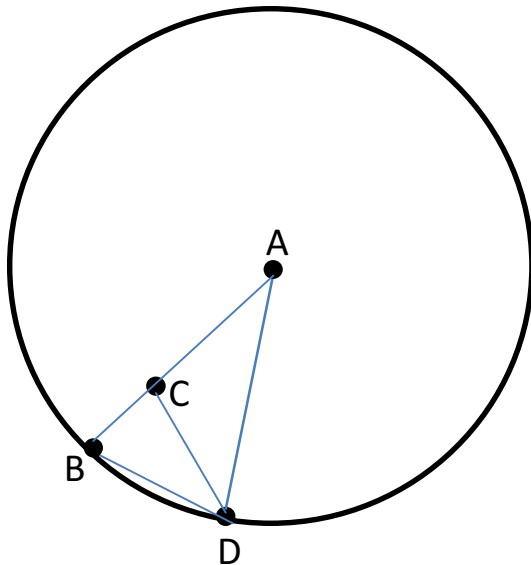
- 16 Propositions
- Deals with figures inscribed in and circumscribed about circles
 - Triangles, squares, regular pentagons, and regular hexagons
- The last proposition shows how to inscribe a 15-sided regular polygon in a given circle.
- The 15-sided regular polygon is said to have been used in astronomy. The angle of the ecliptic was believed to be $360/15$.



Book 4

Inscribing and circumscribing triangles, squares, and regular polygons in and about circles

– Includes pentagon, hexagon, and 15-gon

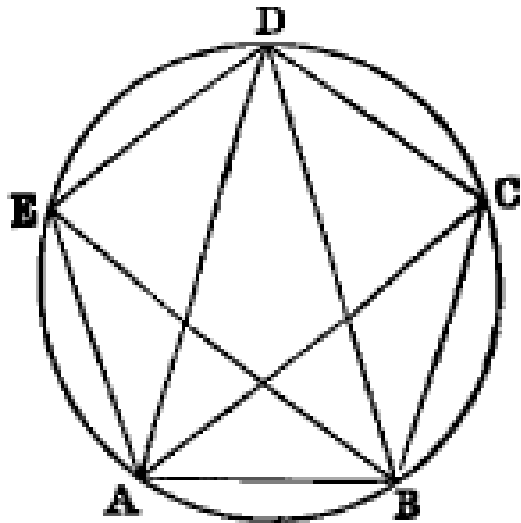


To Construct a regular pentagon, first construct an isosceles Δ with base angle $2x$ vertical angle

- 1) Draw the radius AB
- 2) Determine point C such that $AB/AC = AC/BC$
- 3) Construct $BD = AC$
- 4) From 2 & 3 $AB/BD = BD/BC$
- 5) ΔBDC is similar to ΔBAD
- 6) Δ s BAD, BDC, and DCA are isosceles
- 7) Angle ABD = $2x$ angle BAD

Note: I have not followed the Euclid theorem sequence in the proof of the construction.

Inscribe a Regular Pentagon in a Circle



- 1) Construct the isosceles Δ of the previous slide in the circle (ΔABD)
- 2) Bisect angles ABD and DAB and extend to points E and C
- 3) Figure ABCDE is the inscribed pentagon
- 4) By inscribing a hexagon inside the circle starting at one of the pentagon's vertices a 15 – gon can be constructed.

Why is this interesting?

It took 2000 years until 1796 that Carl Friedrich Gauss proved that the regular 17-gon could be constructed.

Fermat Primes

- Primes of the form $P(n) = 2^{2^n} + 1$
- Only known Fermat primes are 3, 5, 17, 257, and 65537 corresponding to $n = 0, 1, 2, 3,$ and 4. $P(5)$ has 10 digits and is not a prime. No one knows if there are more Fermat primes
- Only constructible regular polygons with a prime number of sides is for the Fermat primes
- Only constructible regular polygons with an odd number of sides is for the product of Fermat primes
 - $3 \times 5 = 15, 3 \times 17 = 51, 5 \times 17 = 85, \dots$

Book V, Theory of Proportion

- Proposition 1 $ma + mb + mc + \dots = m(a + b + c + \dots)$
- Proposition 3 $m(na) = (mn)a$
- Proposition 12 If $a:a' = b:b' = c:c'$ etc. then each ratio equals $(a + b + c + \dots)/(a' + b' + c' + \dots)$
- Proposition 17/18 If $a:b = c:d$, then $(a \pm b):b = (c \pm d):d$
- Proposition 20 If $a:b = d:e$ and $b:c = e:f$, then $a:c = d:f$
- Proposition 24 If $a:c = d:f$ and $b:c = e:f$, then $(a + b):c = (d + e):f$

Book 6

- In book 6 Euclid exploited the theory of proportions developed in book 5 to prove theorems related to similar triangles, parallelograms, and other polygons.
- Proposition 31 is a generalization of the Pythagorean theorem and states that if similar figures are constructed on the two arms and hypotenuse of a right triangle the sum of the area of the figures on the sides equals the area of the figure on the hypotenuse.

Books 7, 8, & 9

- Books 7, 8, and 9 treat the properties of whole numbers and the ratios of whole numbers.
- Numbers are represented as line segments and the product of two numbers as a rectangle, but the arguments do not depend on geometry.
- The statements and proofs are verbal as opposed to the modern symbolic form.
- In these three books as in other books, Euclid assumes facts that he does not state explicitly
- If A divides (evenly into) B and B divides C, then A divides C. Also, if A divides B and divides C, it divides $B + C$ and $B - C$.

Book 7 – Definitions (Think Geometry)

1. A unit is that by virtue of which each of the things that exist is called one.
2. A number is a multitude composed of units.
3. A number is a part of a number, the less of the greater, when it measures the greater.
11. A prime number is that which is measured by an unit alone
12. Numbers prime to one another are those which are measured by an unit alone as a common measure.
16. And, when two numbers having multiplied one another make some number, the number so produced is called plane, and its sides are the numbers which have multiplied one another.
22. A perfect number is that which is equal to its own parts.

Book 7, Theory of Numbers

- **Proposition 1** Two unequal numbers (being) laid down, and the lesser being continually subtracted, in turn, from the greater, if the remainder never measures the (number) preceding it, until a unit remains, then the original numbers will be prime to one another. (example 25 & 7)
- **Proposition 2** To find the greatest common measure (GCM) of two given numbers (which are) not prime to one another.
 - Proceed as in proposition 1. Let the numbers be A and B with $B < A$. If B divides A , it is the GCM. If not then calculate $C = A - B$. If C divides B , it is the GCM. If not subtract the smaller of B & C (B') from the larger (A') and if the difference (C') divides B' , it is the GCM. If not, continue until it does. The value of C'^N that divides B'^N is the GCM (example 805 & 385)

Book 7

- P 30 (Euclid's Lemma)
 - If two numbers, multiplied by one another make some number, and any prime number measures the product, then it also measures one of the original numbers.
 - Used in some of the proofs of the Fundamental Theorem of Arithmetic that states any integer greater than 1 can be written as a unique product (up to the ordering of the factors) of prime numbers.
- P 31
 - Any composite number is measured by some prime number

Book 8, Geometric Progressions

- To Euclid a geometric progression is a set of numbers in continued proportion, that is,
 - $a:b = b:c = c:d = d:e = \dots$
 - In our language we would say that $a/b = b/c = c/d = d/e \dots =$ a constant (r)
 - Then, $a = br, b = cr, c = dr, d = er, \dots$
 - $b = a/r,$
 - $c = b/r = a/r^2$
 - $d = c/r = a/r^3$
 - $e = d/r = a/r^4$
- This is a geometric progression

Book contains 27 propositions that are not too complicated when using modern notation.

Book 9, Number Theory

- **Proposition 14** This proposition states that the least common multiple of a set of prime numbers is not divisible by any other prime.

- Also related to the Fundamental Theorem of Arithmetic

- **Proposition 20** Prime numbers are more than any assigned multitude of prime numbers. (There are an infinite number of primes.)

Definition Perfect Number – a number equal to the sum of its own factors.

- Examples: $6 = 1 + 2 + 3$, $28 = 1 + 2 + 4 + 7 + 14$

- Next are: 496, 8128, ...

- **Proposition 36** If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect.

- If $2^p - 1$ is a prime number, then $(2^p - 1) 2^{p-1}$ is a perfect number.

Perfect Numbers

- If $2^p - 1$ is a prime number, then $(2^p - 1) 2^{p-1}$ is a perfect number.
- Prime numbers of the form $2^p - 1$ have come to be called *Mersenne primes* named in honor of Marin Mersenne (1588-1648), one of many people who have studied them.
 - It is necessary for p to be a prime for $2^p - 1$ to be a prime, but it is not sufficient.
- The search for more Mersenne primes, and therefore more perfect numbers, continues. It is not known if there are infinitely many or finitely many even perfect numbers. Mersenne primes are scarce, but more continue to be found. As of 2014 there are 48 perfect numbers.

Mersenne Primes

- Many early writers felt that numbers of the form $2^n - 1$ were prime for all primes n .
- In 1536 Regius showed $2^{11} - 1$ was not a prime.
- By 1603 Cataldi verified that $2^{17} - 1$ & $2^{19} - 1$ were both prime and stated that $2^n - 1$ was also prime for 23, 29, 31, and 37.
- In 1640 Fermat showed Cataldi was wrong about 23 and 37.
- In 1738 Euler showed Cataldi was wrong about 29 and later showed he was right about 31
- In 1644 Mersenne stated that numbers $2^n - 1$ was prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127$ and 257 and were composite for all positive integers $n < 257$.
- Mersenne was wrong but got his name attached to these numbers.

Book 10

- Book 10 of the Elements was, before the advent of early modern algebra, the most admired – and the most feared.
- It is concerned with a systematic classification of incommensurable line segments of the forms

$$a \pm \sqrt{b} \quad \sqrt{a} \pm \sqrt{b} \quad \sqrt{a \pm \sqrt{b}} \quad \sqrt{(\sqrt{a} \pm \sqrt{b})}$$

where a and b , when the same dimension, are commensurable.

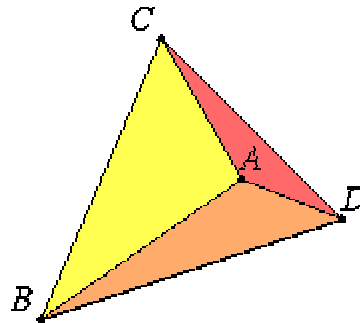
- Book 10 contains 115 propositions – more than any other - most of which contain geometrical equivalents of what we now know arithmetically as surds.
- Some of the work by Eudoxus on the Theory of Exhaustion appears in book 10.

More Book 10

- Definitions
 - D1 Those magnitudes are said to be *commensurable* which are measured by the same measure, and those *incommensurable* which cannot have any common measure.
 - D2 Straight lines are *commensurable in square* when the squares on them are measured by the same area, and *incommensurable in square* when the squares on them cannot possibly have any area as a common measure.
 - D3 With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called *rational*, and those straight lines which are commensurable with it, whether in length and in square, or in square only, *rational*, but those that are incommensurable with it *irrational* .

Books 11

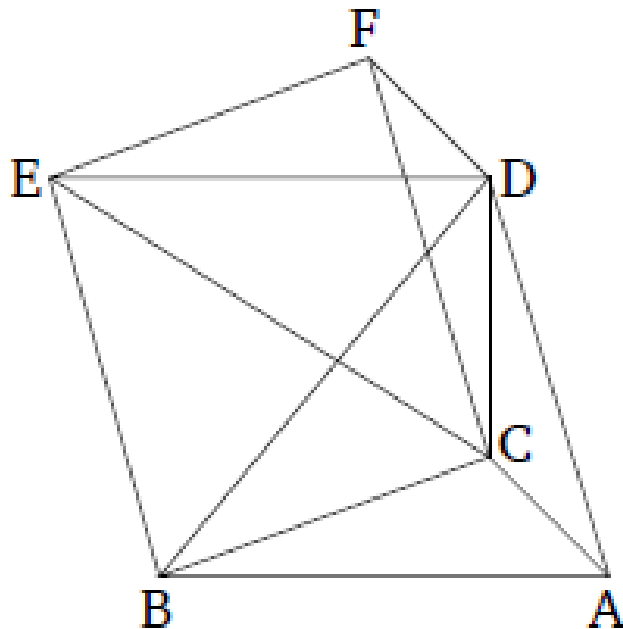
- Book 11 is similar to Books 1 –6 on plane geometry but applied to 3 dimensions – in particular parallelepipeds
- 28 Definitions and 39 Propositions
- Definitions include parallel planes, similar solid figures, solid angle, pyramid, prism, sphere, cone, cylinder, cube, the regular octahedron (8 equal and equilateral triangles), the regular icosahedron (20 equal and equilateral triangles), the dodecahedron (12 equal, equilateral and equiangular pentagons) and other figures.
- D14 When a semicircle with fixed diameter is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a sphere.
- P21 Any solid angle is contained by plane angles whose sum is less than four right angles.



Book 12

- Contains 18 propositions on areas and volumes, particularly of curvilinear figures and figures bounded by surfaces.
- The dominant idea of the book is the method of exhaustion (Eudoxus).
- P1 Similar polygons in circles are to one another as the squares **on** their diameters
- P2 Circles are to one another as the squares on their diameters
- P12 Similar cones and cylinders are to one another in the triplicate ratio of the diameters of their bases .
- P 17 Given two spheres about the same center, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.
- P18 Spheres are to one another in triplicate ratio of their diameters.

Book 12, Volume of a Pyramid



- Proposition 5

Pyramids which are of the same height, and have triangular bases, are to one another as their bases.

- Proposition 7

Any prism having a triangular base is divided into three pyramids having triangular bases (which are) equal to each other.

$$P(\text{BCEvD}) = P(\text{ECFvD}) = P(\text{ABDvC})$$

Volume of a pyramid is $\frac{1}{3}$ volume of a prism with same base and altitude

Cones by method of exhaustion

Book XIII

- The object of Book XIII is to construct each of the 5 regular solids inside of a circumscribing sphere.
 - Tetrahedron - 4 equilateral triangles
 - Octahedron - 8 equilateral triangles
 - Icosahedron - 20 equilateral triangles
 - Cube - 6 squares
 - Dodecahedron - 12 equilateral pentagons

<http://aleph0.clarku.edu/~djoyce/java/elements/bookXIII/bookXIII.html>

The Elements

- The first substantial source of mathematical knowledge.
- Used by all succeeding generations .
- The notion of proof and the logical ordering of propositions were learned by studying it.
- Although the form of presentation of the individual propositions is not original to Euclid, the form of the presentation of the entire work – the statement of all axioms at the outset, the explicit statement of definitions, and the orderly chain of postulates – is his own.
- Euclid's choice of definitions, postulates and common notions is remarkable. From a small set he was able to prove hundreds of postulates, and many of them deep ones.

In Closing

- Euclid's *Elements* form one of the most beautiful and influential works of science in the history of humankind. Its beauty lies in its logical development of geometry and other branches of mathematics. It has influenced all branches of science but none so much as mathematics and the exact sciences. The *Elements* have been studied for over 2000 years in many languages starting, of course, in the original Greek, then in Arabic, Latin, and many modern languages.

From David E. Joyce, Professor of Mathematics and Computer Science, Clark University

References

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