The History of Mathematics
From the Egyptians to Archimedes

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OLLI Winter 2011
Egyptian and Babylonian

• 3000 BC to 260 AD
• Essentially empirical
• Early number systems
• Simple arithmetic, practical geometry
• Egyptian papyri & Babylonian cuneiform tablets
  – Mathematical tables
  – Collections of mathematical problems
Hieroglyphic & Hieratic Numerals

Egyptian hieratic numerals (mathematical papyrus, c. 1600 BC)

<table>
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</table>
Hieratic & Hieroglyphic

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\text{units} & I & II & III \\
\text{tens} & \wedge & \wedge & \wedge \\
\text{hundreds} & \text{male figure} & \text{female figure} & \text{male figure} \\
\end{array}
\]

\[
\begin{array}{c}
\frac{24}{53} \frac{53}{77} \\
\frac{37}{46} \frac{46}{83} \\
\frac{259}{376} \frac{376}{635} \\
\end{array}
\]
Egyptian Arithmetic

• The Egyptians could add, subtract, multiply and divide
• Multiplication of whole numbers used the method of doubling

Example: 57 x 117

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Primary Egyptian Sources

• Rhind Mathematical Papyrus (RMP)
  – About 1650 BC from writings made 200 years earlier (18 ft x 13 in)
  – The Recto Table and 84 (87) mathematical problems

• Moscow Mathematical Papyrus –
  – 1850 BC, (18 ft x 1.5 to 3 in)
  – 25 problems
The Recto Table

- The division of 2 by the odd numbers 3 to 101 with the answers expressed as the sum of unit fractions
- The entrance into the knowledge of all existing things and all obscure secrets. This book was copied in the year 33, in the 4th month of the inundation season, under the majesty of the king of Upper and Lower Egypt, A-user-Re (Aweserre Apopi), endowed with life, in likeness to writings of old made in the time of Upper and Lower Egypt, Ne-ma-et-Re (Nemare Ammenemes III). It is the scribe Ah-mose who copies this writing.
Egyptian Fractions & Algebra

- Except for 2/3 and possibly 3/4 the Egyptian arithmetic notation only permitted fractions with unity in the numerator.
- $2/n$ equivalents in the RMP Recto table:
  
  | 5: 3 15 | 7: 4 28 | 9: 6 18 | 11: 6 66 | 13: 8 52 104 |
  | 15: 10 30 | 17: 12 51 68 | 19: 12 76 114 | 21: 14 42 |
  | 23: 12 276 | 25: 15 75 ... | 97: 56 679 776 ... |

- Could multiply and divide mixed numbers.
- Simple Algebra
  - A quantity and its seventh, added together give 19. What is the quantity. (Method of single false position)
  - The sum of the areas of two squares is 100. Three times the side of one is four times the side of the other. Find the sides of the squares.
  - Arithmetic series and geometric series
Rhind Papyrus
Problem 24 of RMP
Egyptian Geometry

• Area and perimeter
  – Rectangle, triangle, trapezoid, arbitrary quadrilateral, circle
  – Rule for arbitrary quadrilateral was incorrect
  – Area of circle given by the area of a square with side $8D/9$ where $D$ is the circle diameter. Implies $\pi \approx 3.16$.
  – Egyptians knew that the ratio area/perimeter for a circle and a circumscribed square are equal. (some debate)
  – May have known the surface area of a sphere

• Volume
  – Pyramid, truncated pyramid
Egyptian Mathematics

• The Egyptian papyri show practical techniques for solving everyday problems
• The rules in the papyri are seldom motivated and the papyri may in fact only be manuals for students
• However, they demonstrated a solid understanding of the operations of addition, subtraction, multiplication and division and enough geometry for their needs
• There was little change in their approach for at least 1000 years
Mesopotamian Mathematics

- Babylonian civilization (2000 BC – 600 BC) replaced Sumerian and Akkadian civilizations
- Base 60 positional number system
- Numbers system built on symbol for 1 and 10
- No zero and no character to indicate a position was vacant until 300 BC
- More general treatment of fractions than Egyptians
- Great advance was that their number system was extended to fractions and positional notation.
Symbols for numbers 1 – 59

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Number System Examples

- Use ‘I’ for the Babylonian one and ‘<‘ for the Babylonian 10
- <<II = 22, <<<IIII = 34
- << << = 20*60 + 20 = 1220
- << << = 20*60^2 + 20*60 = 73200
- II II <I = 2*60 + 2 + 11/60
- I II could be 62 or 3602

In approximately 300 BC the Babylonians added a symbol like “” to show if a position was empty. It was only used for intermediate position and not end positions (no true zero)
- I “ II means 1*60^2 + 0*60 + 2 or 60 + 0 + 2/60
Babylonians could multiply in their base 60 system the same way we multiply in our base 10 system

\[
\begin{array}{ccc}
139 & 2,19 \\
x72 & 1,12 & 12 \times 19 = 3,48 \text{ (from table)} \\
278 & 27,48 \\
973 & 2,19 \\
10008 & 2,46,48 & = 2 \times 60^2 + 46 \times 60 + 48 = 10008
\end{array}
\]
Other Features

• Developed an accurate process for finding square roots
  divide & average  \( \sqrt{2} = 1.41421296 \)
  correct    \( \sqrt{2} = 1.41421356 \)

• Constructed tables of squares to assist in multiplication
  \( xy = ((x + y)^2 - (x - y)^2)/4 \)

• Able to solve quadratic equation and linear equations in two variables

• Solved cubic equations using tables and interpolation

• Arithmetic and geometric progressions
Quadratic Equation Prescription

Problem Statement: Find two numbers with sum 20 and product 91

1) Find half of the sum: 10
2) Subtract the product from the square of result of step 1: 100 – 91 = 9
3) Take the square root of the result of step 2: 3
4) Add and subtract the result of step 3 to the result of step 1: 10 + 3 = 13 and 10 – 3 = 7
5) The numbers are 13 and 7
Pythagorean Triples

A Pythagorean triple is a set of three integers such that $a^2 + b^2 = c^2$

Example: $3^2 + 4^2 = 5^2$

Plimpton 322 contains 15 sets of three numbers. The 2\textsuperscript{nd} and 3\textsuperscript{rd} numbers are part of triplets

Example (2\textsuperscript{nd} line): $3456^2 + 3367^2 = 4825^2$

The 1\textsuperscript{st} number is a fraction related to two of the sides of the triangle
Generally believed that the Babylonians knew the Pythagorean Theorem
Geometry not a Babylonian Strength

For practical problems they calculated the area of a circle by $A = \frac{C^2}{12}$ where $C$ is the circumference. This implies a value $\pi = 3$. Tablets found in Susa included the ratio of the perimeter of a hexagon to the circumference of the circumscribed circle and implied a value of $\pi = 3.125$. Babylonian geometry was a collection of rules for the areas of plane figures and the volumes of simple solids.
Greek/Hellenistic Timeline

- **Classical Period (500-336 BC)** - Classical period of ancient Greek history, is fixed between about 500 B.C., when the Greeks began to come into conflict with the kingdom of Persia to the east, and the death of the Macedonian king and conqueror Alexander the Great in 323 B.C. In this period Athens reached its greatest political and cultural heights: the full development of the democratic system of government under the Athenian statesman Pericles; the building of the Parthenon on the Acropolis; the creation of the tragedies of Sophocles, Aeschylus and Euripides; and the founding of the philosophical schools of Socrates and Plato.

- **Hellenistic Period (336-146 BC)** - period between the conquest of the Persian Empire by Alexander the Great and the establishment of Roman supremacy, in which Greek culture and learning were pre-eminent in the Mediterranean and Asia Minor. It is called Hellenistic (Greek, Hellas, "Greece") to distinguish it from the Hellenic culture of classical Greece.
Early Greek Number System

• Acrophonic (Attic, Herodianic)

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<th>Χ</th>
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~ 600 BC to ~ 300 BC

Similar to Roman Numerals

Denotes drachma
Later Greek Number System

• Ionic (Alphabetical)

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εχοη = 5678
Fractions

• Acrophonic
  – Essentially the same as the Egyptians

• Ionic
  – Used apostrophe to signify fraction
    \[ \gamma' = \frac{1}{3} \quad \lambda\beta' = \frac{1}{32} \]
    \[ \bar{i} \ o\alpha' = \frac{10}{71} \ (Archimedes) \]

• Addition, subtraction, multiplication and division done the same way we do it
Greek Mathematics

Roughly 600 BC to 300 AD
We will cover 600 BC to 200 BC
How do we know about Greek mathematics?

• Initially knowledge was passed from teacher to student orally.
• Probably around 450 BC, chalk boards and wax tablets were introduced for non-permanent work.
• Papyrus rolls were used for permanent records but new copies were required frequently.
• In about 300 BC, Euclid’s Elements was completed and it was so comprehensive and of such quality that all older mathematical texts became obsolete.
• In 2nd century AD, books of papyrus appeared and became the main form in the 4th century. Also, vellum (animal skin) was introduced.
How do we know about Greek mathematics?

- If the person copying the “Elements” was a mathematician, material not in the original text may have been added.
- The oldest surviving complete copy of the Elements is from 888 AD probably based on a version with commentary and additions produced by Theon of Alexandria in the 4th century AD.
- In addition to the 888 AD document there are numerous fragments some dating to as early as 225 BC.
- Some surviving texts exist that were written after 888 AD that were based on versions of the elements earlier than 888 AD.
- First print edition in Venice in 1482.
- For the complete story see the article “How do we know about Greek mathematics” on the Mac Tutor History of Mathematics.
Greek Mathematics – Thales of Miletus

• Thales of Miletus 624 BC – 547 BC (Turkey)
  – None (if there were any) of his writings survive
  – Name appears in the writings of others years later
  – The first of the seven sages of antiquity
  – A pupil of the Egyptians
  – Credited with five theorems of elementary geometry
    • A circle is bisected by any diameter
    • The base angles of an isosceles triangle are equal
    • The angles between two intersecting straight lines are equal
    • Two triangles are congruent if they have two angles and a corresponding side equal
    • An angle in a semicircle is a right angle
Greek Mathematics – Pythagoras of Samos

• Pythagoras (569 BC – 475 BC)
  – Another very obscure figure. Biographies written in antiquity all lost
  – Founded a school that was both communal and secret
  – Difficult to separate history and legend
  – The Pythagoreans examined the basic principles of mathematics and gave it an intellectual structure.
  – Major mathematical influence on Pythagoras likely to have been Anaximander, a pupil of Thales, and the Egyptians
  – The Pythagoreans taught that the purpose of life was to purify the soul and body. To reach purification one had to discover the “harmonies” of the cosmos – and scientific (mathematical) inquiry was the vehicle with which to find them.
One of the scientific achievements of Pythagoras was the discovery of the mathematical order in the musical scale and the harmonies so produced. It is believed that he experimented with string instruments and discovered that two tones sound well together when the ratios of their frequencies can be expressed by the use of small numbers and the smaller the numbers the better is the consonance.

Table from “Science & Music” by Sir James Jeans
Pythagorean Scale

<table>
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<th>Pythagorean frequency ratio</th>
<th>Pythagorean interval</th>
<th>Equal temperament frequency ratio (p. 177)</th>
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<td>Tone</td>
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Table from “Science & Music” by Sir James Jeans
All is Number

• Boethius (~ 500 CE) tells us that Pythagoras investigated the relation between the length of a vibrating string and the musical tone it produced. If a string was shortened to \( \frac{3}{4} \) of its original length, then what is called the fourth of the original tone was heard; if shortened to \( \frac{2}{3} \), the fifth was heard; and if shortened to \( \frac{1}{2} \), the octave. The string lengths were proportional to 12, 9, 8, and 6.

• The Pythagorean school was able to find many interesting relations between these number including the fact that a cube has 6 faces, 8 vertices, and 12 edges. They were able to convince themselves that since these combinations of string lengths produced sounds that harmonize, the numbers themselves essentially caused it. The Pythagoreans extrapolated to the natural numbers being fundamental to natural science.

• *This was bad science but really good for number theory*
Pythagorean “Arithmetica”

- Pythagorean triplets – natural numbers that satisfy
  \[ a^2 + b^2 = c^2 \]
- Pythagoreans knew that
  \[ (m^2 - 1)^2 + (2m)^2 = (m^2 + 1)^2 \]
produced triplets. For example for \( m = 2 \), the triplet is 3, 4, 5. However the formula does not produce all triplets. Euclid solves the general problem is his Elements.
More “Arithmetica”

- Perfect numbers – the number is the sum of its divisors.
  - \(6 = 1 + 2 + 3\)
  - \(28 = 1 + 2 + 4 + 7 + 14\)
  - \(496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248\)
  - \(8128\)
  - \(33550336\)

  The search for perfect numbers continues to today. No odd perfect numbers have been found.

- Friendly numbers – pairs of numbers such that each is the sum of the divisors of the other –

  The first friendly pair is 220 & 284
  The secondly pair is 17,296 and 18,416 and was independently discovered by al-Banna (Arab 1256 – 1321) and by Fermat in 1636
  In the 1970s the phrase “friendly numbers” started to be used differently
Figurate Numbers

Triangular Numbers  1, 3, 6, 10, 15 ...  
Square Numbers  1, 4, 9, 16, 25 ...  
The sum of two consecutive triangular numbers is a square number

The sum of any number of consecutive odd integers, starting with one, is a perfect square
More “Arithmetica”

• The Pythagoreans studied prime numbers, progressions, and ratios and proportions. They understood that certain sums could be easily calculated.
  \[1 + 2 + \ldots + n = \frac{n}{2} (n + 1)\]

• Numbers to Pythagoreans were whole numbers only. The ratio of two whole numbers was not a fraction and therefore another kind of number. Actual fractions were employed in commerce.

• The most developed part of the *arithmetica* was the theory of even and odd
  – The sum of two even numbers is even
  – The product of two odd numbers is odd
  – When an odd number divides an even number, it also divides its half
Pythagorean School Geometry

• In the 5th century BC geometry, the theorem/proof logic took root and a system of theorems was developed where theorems were proved based on theorems previously proved.
• Studied triangles, parallel lines, polygons, circles, spheres and regular polyhedra
• They knew that the sum of the angles of a triangle equals two right angles
• Worked on a class of problems known as “application of area”
  – Example – Construct a polygon equal in area to a given one and similar in shape to another given one.
Pythagoreans' Symbol

\[
\frac{\text{red}}{\text{green}} = \frac{\text{green}}{\text{blue}} = \frac{\text{blue}}{\text{magenta}} = \varphi.
\]

Golden Ratio
Golden ratio and Golden Rectangle
For a right triangle with arms $a$ and $b$ and hypotenuse $c$, then the area of the square $A$ constructed on $a$ plus the area of the square $B$ constructed on $b$ equals the area of the square $C$ constructed on $c$. 
Proof of Pythagorean Theorem

The areas of the two large squares are equal. The area of the square on the left is 4 triangles + square A + square B. The area of the square on the right is 4 triangles + square C. Since all eight triangles are identical, The area of A + the area of B = the area of C
Rational Numbers

• The concept of integers comes from counting objects
• Life requires us to measure quantities such as length, speed, weight, time, ...
• To satisfy these measurements fractions (ratios) are required. No matter how accurately it is necessary to measure something, that measurement can always be expressed as a fraction of the unit of measure.
• Early mathematicians thought of numbers as points on a line and believed that all points could be expressed as fractions.
Irrational Numbers

• From the Pythagorean theorem it follows that for an isosceles right isosceles triangle with arm equal to 1 the hypotenuse equals $\sqrt{2}$. The Pythagoreans tried to find two natural numbers whose ratio was $\sqrt{2}$, but failed.

• The discovery of “irrational” ratios is attributed to Hippasus of Metapontum (5th cent. B.C.)

• After the $\sqrt{2}$ was revealed to be irrational, the Pythagoreans according to a legend killed Hippasus, not willing to believe this fundamental number could fail to be a ratio of integers.

• In Plato’s Republic it is stated that $\sqrt{50}$ is irrational.
• Proof that $\sqrt{2}$ is irrational
  – $a/b = \sqrt{2}$ where both $a$ and $b$ are integers and the ratio $a/b$ has been reduced to lowest terms. This means that $a$ and $b$ cannot both be even.
  – $a^2 = 2 \cdot b^2$
  – $a$ must be even since if it were odd then $a^2$ would be odd.
  – If $a$ is even it can be written as $2c$ implying $4c^2 = 2 \cdot b^2$ or $2c^2 = b^2$
  – By the same argument as above $b$ must be even.
  – The assumption that $\sqrt{2}$ is rational leads to a contradiction.
Platonic Solids


Pythagoras

- Tetrahedron – 4 equilateral triangles
- Cube (hexahedron) – 6 squares
- Dodecahedron – 12 equilateral pentagons

• Theaetetus (417 BC – 369 BC)
  - Octahedron – 8 equilateral triangles
  - Icosahedron – 20 equilateral triangles

• In book XIII of Euclid’s elements it is shown that there are only five convex regular polyhedra

• So why are these called the Platonic solids?
Platonic solids

- Octahedron (Air)
- Cube (Earth)
- Tetrahedron (Fire)
- Dodecahedron (the Universe)
- Icosahedron (Water)
Three Famous Geometry Problems of Antiquity

• These three problems were to be solved using a straight edge (not a ruler) and compass
  – Squaring the circle (Anaxagoras, 500 BC – 428 BC)
  – Doubling the cube (before Hippocrates of Chios, 470 BC – 410 BC)
  – Trisecting an angle (before Hippocrates of Chios)
Quadrature of Lunes

The area of the semicircle ACE equals the area of the quarter circle ADC. Hence the area of the lune ACEF equals the area of triangle ADC.

The fact that a shape that appears to be related to a circle can be shown to be equal in area to a triangle led folks to believe that the circle could be squared.
Famous Problems

• Squaring the circle
  – Squaring the circle involves constructing the square root of $\pi$ which was shown to be impossible in 1882 when Ferdinand von Lindemann proved that $\pi$ is transcendental.

• Doubling the cube (Delian problem) & trisecting an angle
  – Both these problems require finding the cube root of a quantity. In 1837 Pierre Wantzel showed the problems are unsolvable by compass and straightedge construction.
  – There are certain angles that can be trisected, e.g., 90 degrees but in general it cannot be done.
Beginning of the “Elements”

• Hippocrates of Chios
  – wrote the first Elements around 430 BC.
  – After Hippocrates at least four other Elements were written each improving on the previous versions
  – Euclid wrote his Elements in about 325 BC

• Hippocrates of Cos (460 BC – 375 BC)
  – father of rational medicine
Eudoxus of Cnidus (408 – 355 BC)

• Some consider him to be the greatest of the ancient mathematicians, second only to Archimedes

• Resolved the difficulty in comparing rational and irrational numbers

• Put the method of exhaustion on a rigorous basis
  – Similar to the limit concept of calculus
  – Proved the volume of a pyramid (cone) equals 1/3 vol. prism (cylinder) with same base and equal height
  – Area of circles go as the square of their diameters and the volume of spheres go as the cubes of their diameters
Euclid’s Elements (300 BC)
The greatest mathematical textbook of all time

<table>
<thead>
<tr>
<th>Book</th>
<th>Subject</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>The fundamentals of geometry: theories of triangles, parallels, and area.</td>
</tr>
<tr>
<td>II</td>
<td>Geometric algebra.</td>
</tr>
<tr>
<td>III</td>
<td>Theory of circles.</td>
</tr>
<tr>
<td>IV</td>
<td>Constructions for inscribed and circumscribed figures.</td>
</tr>
<tr>
<td>V</td>
<td>Theory of abstract proportions.</td>
</tr>
<tr>
<td>VI</td>
<td>Similar figures and proportions in geometry.</td>
</tr>
<tr>
<td>VII</td>
<td>Fundamentals of number theory.</td>
</tr>
<tr>
<td>VIII</td>
<td>Continued proportions in number theory.</td>
</tr>
<tr>
<td>IX</td>
<td>Number theory.</td>
</tr>
<tr>
<td>X</td>
<td>Classification of incommensurables.</td>
</tr>
<tr>
<td>XI</td>
<td>Solid geometry.</td>
</tr>
<tr>
<td>XII</td>
<td>Measurement of figures.</td>
</tr>
<tr>
<td>XIII</td>
<td>Regular solids.</td>
</tr>
</tbody>
</table>
Ptolemy I (323 – 285 BC)

• Upon the death of Alexander the Great in 323 BC, the throne of Egypt fell to Ptolemy one of Alexander’s trusted commanders

• It is said that Ptolemy once asked Euclid if there was in geometry any shorter way than that of the Elements, and he replied that there was no royal road to geometry.
Euclid (Books 1 – 3)

• Book 1
  – Deals with basic definitions of points, lines and plane figures and related simple plane properties

• Book 2
  – Using the properties of rectangles and squares, introduces geometric algebra including solving the quadratic equation and the law of cosines. Shows how to convert any given polygon into a square of equal area.

• Book 3
  – Various properties of circles including Thales theorem and the properties of tangents
Euclid Book II (Geometric Algebra)

- Example: Given a line segment AB, find BC so that the area of the rectangle AD equals the area of square RT given that CD = BC.

What would we do?

AB = a, BC = x, CD = x, RS = s

x (a + x) = s^2, Solve for x.

Example: a = 6, area of RT = 16.

x^2 + 6x – 16 = 0

(x + 8)(x – 2) = 0; x = 2, x = -8
Example

AB is original line segment.
M is the midpoint of AB.
MN is the side of the given square.
Construct the line segment MC equal to AN.
BC is the solution.

Check to see if correct
The sq on AM plus the sq on MN equals the sq on AN. The sq MP equals on AN. The sq on AM equals the sq LE. Sum of blue areas = sq on MN. AMGF = EDPO
Sq on MN = ACDF
Book 4

Inscribing and circumscribing triangles, squares, and regular polygons in and about circles

– Includes pentagon, hexagon, and 15-gon

To Construct a regular pentagon, first construct an isosceles \( \triangle \) with base angle 2x vertical angle

1) Draw the radius \( AB \)
2) Determine point \( C \) such that \( \frac{AB}{AC} = \frac{AC}{BC} \)
3) Construct \( BD = AC \)
4) From 2 & 3 \( \frac{AB}{BD} = \frac{BD}{BC} \)
5) \( \triangle BDC \) is similar to \( \triangle BAD \)
6) \( \triangle s \) BAD, BDC, and DCA are isosceles
7) Angle BDA = 2x angle BAD

Note: I have not followed the Euclid theorem sequence in the proof of the construction.
Inscribe a Regular Pentagon in a Circle

1) Construct the isosceles Δ of the previous slide in the circle (ΔABD)
2) Bisect angles ABD and DAB and extend to points E and C
3) Figure ABCDE is the inscribed pentagon
4) By inscribing an hexagon inside the circle starting at one of the pentagon’s vertices a 15 – gon can be constructed.

Why is this interesting?
It took 2000 years until 1796 that Carl Friedrich Gauss proved that the regular 17-gon could be constructed.
The Theory of Numbers (Books 7 – 9)

• Book 7
  – Definitions: even, odd, prime, relatively prime (no common divisors), composite, perfect (6, 28, 496, 8128, ..) & others
  – Perfect numbers (number equals sum of divisors)
    • 6: divisors 1, 2, 3
    • 28: divisors 1, 2, 4, 7, 14
    • 496: divisors 1, 2, 4, 8, 16, 31, 62, 124, 248

In Book 9 Euclid showed that \( n = 2^{(m-1)}(2^m - 1) \) is perfect if \( 2^m - 1 \) is a prime

  – Greatest Common Divisor (Euclid’s Algorithm)
  – Least Common Multiple
Book 8

• Book 8 deals primarily with geometric series
  – 2, 2x3, 2x3^2, 2x3^3, 2x3^4
    • 2, 6, 18, 54, 162
  – 32, 48, 72, 108, 162, 243
    • The fact that the first and last terms are relatively prime produces interesting consequences discussed by Euclid
Book 9 +

- Book 9 +
  - No prime can divide a product of other primes
    - A partial version of the fundamental theorem of arithmetic
  - There are an infinite number of primes
  - Sum of a geometric series
  - Euclid’s rule for perfect numbers
  - Book X. Classification of incommensurables.
  - Book XI. Solid geometry.
  - Book XII. Measurement of figures.
  - Book XIII. Regular solids.

- Euclid’s other books
  - Data, lost books (Pseudaria, Porisms, Conics, and Surface – Loci)
Archimedes
Archimedes

• Appears as a historical figure in 216 - 212 BC during the siege and capture of Syracuse
• Having got ready their blindages, missiles, and other siege material, they were in high hopes owing to their large numbers that in five days their works would be much more advanced than those of the enemy, but instead they did not reckon with the ability of Archimedes, or foresee that in some cases the genius of one man accomplishes much more than any number of hands. (Polybius)
• Killed when the city was sacked (age 75)
Archimedes and the Burning Mirror
Inventions

Screw Pump

The Archimedes Claw
Archimedes Mathematical Achievements

- Area of a circle (area of rt Δ with base = circumference and altitude = radius)
- Circumference = $2\pi r$, $3 \frac{10}{71} < \pi < 3 \frac{1}{7}$
- Area of a segment of a parabola
- Area of an ellipse
- Volume and surface area of a sphere
- Volumes of various "solids of revolution" obtained by rotating a curve about a fixed straight line.
- Law of the lever, center of gravity, hydrostatics
Archimedes Mathematical Achievements (cont.)

• One of the methods he used to find the areas, volumes and surface areas of many bodies was an early form of integration. This was considered his greatest mathematical invention, leading to the field of Calculus. To determine the area of sections bounded by geometric figures such as parabolas and ellipses, Archimedes broke the sections into an infinite number of rectangles and added the areas together.
Great Mathematicians

Gauss, Newton, Archimedes, Euler, Cauchy, Poincare, Riemann, Cantor, Cayley, Hamilton, Eisenstein, Pascal, Abel, Hilbert, Klein, Leibniz, Descartes, Galois, Mobius, Jacob, Johann and Daniel Bernoulli, Dirichlet, Fermat, Pythagoras, Laplace, Lagrange, Kronecker, Jacobi, Bolyai and Lobatchewsky, Noether, Germain, Euclid, Legendre

http://angelustenebrae.livejournal.com/15908.html?thread=37668
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